THE MATHEMATICAL GAZETTE

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc., F.R.S.
AND
Prof. E. T. WHITTAKER, Sc.D., F.R.S.

LONDON
G. BELL & SONS, LTD., PORTUGAL STREET, KINGSWAY, W.C. 2.
AND BOMBAY

Vol. XIV., No. 199. MARCH, 1929. 3s. 6d. Net.

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Intending members are requested to communicate with one of the Secretaries. The subscription to the Association is 15s. per annum, and is due on Jan. 1st. It includes the subscription to "The Mathematical Gazette."

Change of Address should be notified to a Secretary. If Copies of the "Gazette" fail for lack of such notification to reach a member, duplicate copies can be supplied only at the published price.

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VOL. XIV. MARCH, 1929. No. 199.

The Mathematical Association.

THE Annual Meeting of the Mathematical Association was held at the London Day Training College, on 7th-8th January, 1929.

On Monday, January 7th, the proceedings opened with the transaction of formal business, the presentation and adoption of the Report of the Council * and the Report of the Treasurer, for 1928, and with the re-election, on the nomination of the Council, of Dr. W. F. Sheppard as President for 1929.

The election of Officers and Council for 1929 was then proceeded with. Mr. W. Hope-Jones and Miss E. Wise were elected, on the nomination of the Council, to vacancies caused by the retirement, in compliance with the rules, of Miss M. A. Hooke and Mr. N. M. Gibbins.

The President took the chair and called on Mr. H. G. Forder, B.A., to read his address † on *The Axioms of Geometry*.

This was followed, after the usual interval, by a lecture ‡ on *Modern Mathematical Problems in Aerodynamics*, delivered by Prof. H. Levy, D.Sc., F.R.S.E. (Imperial College of Science and Technology).

On Tuesday, January 8th, the morning was devoted to discussions on *The Choice of Groups by Candidates for School Certificates*, and on *The Use and Abuse of Formulae*. These were opened by Miss E. R. Gwatkin, M.A., and Mr. N. J. Chignell, M.A., respectively.§ The following members took part in the discussions: Mr. W. C. Fletcher, Mr. B. L. Gimson, Mr. W. Hope-Jones, Mr. J. Katz, Mr. F. H. Knight, Prof. E. H. Neville, Miss Punnett, Prof. J. E. A. Steggall, Miss C. Waters.

The President opened the afternoon meeting with an address ‡ on *Variety of Method in the Teaching of Arithmetic*.

After the interval, the chair was taken by Prof. A. Lodge. Prof. J. E. A. Steggall, M.A., F.R.S.E., was invited to deliver the closing address,‡ and took as his subject—*Methods of Voting in Theory and Practice*.

A vote of thanks to the Chairman brought the proceedings to a close.

* *v.* pp. 318-319. † *v.* pp. 321-332.

‡ This address, or a summary by the author, will appear in a later number of the *Gazette*.

§ *v.* pp. 332-344; 344-352.

REPORT OF THE COUNCIL FOR THE YEAR 1928.

DURING the year 1928, 92 new members of the Association have been admitted. The number of members now on the Roll is 1188; of these 7 are Honorary Members, 60 are Life Members by composition, 21 are Life Members under the old rule (now rescinded), and 1100 are ordinary members.

The Council regret to have to record the deaths of Mr. J. D. St. Clair MacLardy, formerly Principal of the Teachers' College, Sydney, and afterwards Chief Examiner in the Department of Education; Professor R. W. Genese, formerly of the University College of Wales, Aberystwyth, who was a Vice-President of the Association and had just completed fifty years of membership; Lieutenant-Colonel Allan J. C. Cunningham, formerly of the Royal Engineers; Captain C. H. Ilott (late R.G.A.), of Rio de Janeiro; Mr. G. D. Tripp, of the King's School, Rochester, who was killed at Chancery Lane Station; Professor F. S. Carey, formerly of the University of Liverpool; Professor G. H. Bryan, Sc.D., F.R.S., formerly of University College, Bangor, and President of the Association for the years 1907 and 1908; and the Rev. W. H. Laverty, Rector of Headley, Hants, who was one of the original members of the Association in 1871.

In May last a meeting was held at Armstrong College, Newcastle, at which it was decided to form a branch of this Association to be called the North-Eastern Branch. Mr. J. Strachan, H.M.I., was elected President; Miss J. Dow, 12 Otterburn Terrace, Newcastle-upon-Tyne, Honorary Treasurer; and Miss M. Waite, The High School, Darlington, and Mr. A. K. Wilson, Dame Allan's School, Newcastle-upon-Tyne, Honorary Secretaries. The new branch (the eleventh branch of the Association to be formed) started well with more than 60 members.

Suggestions having been received from more than one quarter that members of the Association would sometimes be glad to obtain help in the solution of problems, arrangements have been made in order that such assistance may be given. A small Committee has been formed for this purpose, and members wishing for help are asked to communicate with Mr. A. S. Gosset Tanner, of Derby School, Derby.

During the year "The Report on the Teaching of Mathematics in Public and Secondary Schools," originally issued as No. 143 (Dec. 1919) of the *Mathematical Gazette*, has been revised and copies have been distributed to the members of the Association. The Report on "The Teaching of Mathematics in Preparatory Schools," originally issued in 1924, has also been reprinted. The Teaching Committee have been engaged during the year in revising the "Report on the Teaching of Mathematics in Girls' Schools" (1916), and also in drawing up a new "Report on the Teaching of Mechanics," to replace that issued in 1918 as No. 137 of the *Mathematical Gazette*. Considerable progress has been made with

both, and it is hoped that the former will be ready for distribution to members early in 1929.

The Library has been of steady service to members throughout the year. Gifts have been numerous, but other benefactions have been overshadowed by that of the late Professor R. W. Genese, by whose desire all that he retained of his own mathematical library, when he retired from Aberystwyth some years ago, was presented to the Association after his death. Mr. F. Beames has again given invaluable help to the Librarian.

The Council desire to express their thanks to Miss M. A. Hooke and Mr. N. M. Gibbins, who now retire from the Council in compliance with the rules, for the services which they have rendered to the Association during their tenure of office as members of the Council since 1926 and 1925 respectively.

The Council desire again to acknowledge the indebtedness of the Association to Mr. W. J. Greenstreet for the continuance of his work as Editor of the *Mathematical Gazette*. Mr. Greenstreet's name first appears as joint-editor on No. 12 (October 1897), and as sole editor on No. 16 (February 1899). At the close of this year, 1928, he will therefore have completed thirty years' service as sole editor.

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921 [L¹. 5. b.] *Note* 914.

It might be added that the example given in this note is not merely an isolated one. There is a considerable class of problems about normals to the ellipse which are susceptible of good solutions with the help of the hyperbola of Apollonius $c^2xy + b^2\beta x - a^2\alpha y = 0$ and which lead to clumsy work when attacked by crude methods. The point is illustrated in Askwith, Arts. 170, 171, where after the crude solution has been given it is pointed out that the hyperbola leads to the result. The same thing happens in C. Smith's *Conics*; at the end of a very offensive piece of Trigonometry (Art. 139) reference is made to the better method in Art. 199. Have not writers in the past made the mistake of elaborating the work about conics too far before providing the necessary tools? In this case what is wanted is the principle that a conic through the points of intersection of $s=0$ and $s'=0$ has an equation $s=ks'$. A principle of such importance ought not to be relegated to the last few chapters of a serious work on Analytical Geometry. A. R.

922. [K¹. 6. a.] *Note on* § 14, *p.* 295, *Gazette, Jan.* 1929.

In the article in the January *Gazette* on Homogeneous Line Coordinates a remark is relegated to § 14 which deserves to appear at the beginning.

For from the fact that unit forces along lines (x, y, z), (x_2, y_2, z_2) are equivalent to $(x_1/2\Delta, \ldots)$ and $(x_2/2\Delta, \ldots)$ along the sides, follows the result of § 3,

$$\cos\phi_{12} = \text{the projected product of the two vectors} \quad \frac{[\Sigma x_1x_2 - \Sigma(y_1z_2 + y_2z_1)\cos A]}{4\Delta^2}.$$

It is from the vector point of view, too, that the result of § 14 itself is best seen. A. R.

THE TEACHING OF GEOMETRY IN SCHOOLS.

MEMBERS of the M.A. who are familiar with this Report will share the gratification of the Council on learning that the large second edition, issued in 1925, is already exhausted.

Opportunity has been taken to correct one slip (p. 62, l. 26, *for* $\angle$s *read* $\triangle$s), and to substitute a quotation from Lagrange for a common perversion: the second sentence on p. 23 now runs:

The much misquoted words "On ne trouvera point de figures dans cet ouvrage" expressed succinctly the purpose of the *Méchanique Analitique*, and they may be held, although this was not their author's intention, to embody a doctrine which even in geometry is an excellent if a severe one for an advanced treatise, but no one would prescribe such a principle for school use.

Otherwise the text of the Report is unchanged.

For the benefit of readers of the *Gazette* who do not know the Report, we may say that it is the most substantial work on teaching for which the Association has been responsible since the enthusiastic days of the A.I.G.T. The result of eighteen months' work by a special committee, it was published in 1923, a limp-bound volume of 74 pages, price two shillings; the changes in the second edition were as slight as those now made in the third edition. In addition to sections on general principles, on the stages into which a course seems naturally to be divided, and on the question of a standard sequence, the Report includes a collection of useful notes, and a section which, to quote the Introduction, "aims at providing something that to most teachers has hitherto been inaccessible, namely, a careful discussion of the principal difficulties encountered in the attempt to keep a course in geometry free from arguments that even the commonsense logic which is appropriate to the schoolroom must recognise as defective." Admitting that parts of this section make more serious demands on the reader than the 'average teacher' is expected to entertain, the Committee expressed in the Introduction a hope that "among real teachers there may be some to welcome the implied assumption that they are actually interested in geometry and willing to think carefully"; the need for a third edition is the authors' complete vindication.

GLEANINGS FAR AND NEAR.

642. Reuben Burrows published, in M,DCC,LXXIX, *A Restitution of the Geometrical Treatise of Apollonius Pergaeus on Inclinations: Also the Theory of Gunnery, or the Doctrine of Projectiles in a non-resisting Medium.* The author states that he began this and finished it in less than a week. Of Horsley's *Apollonii Pergaei Inclinationum Libri Duo*, published in 1770, Burrow has a low opinion: "As to the work of Dr. Horsley, it is split into such an infinity of different cases, frittered into so many divisions and sub-divisions, and treated besides in a manner so closely bordering on Algebra, that it does not appear to have the least similarity to any of the genuine productions of Apollonius . . . not to mention the inelegance of his method, his virulent remarks, and his arrogant and contemptuous expressions against former writers."

But Burrows seems to have been unstable as water and to have caught something of the Bishop's intolerance, for a couple of dozen pages further on, in the Preface to *The Theory of Gunnery*, he concludes a reference to "the learned Mr. Brown's excellent translation of Euler's *Gunnery*," and "that elegant piece on Projectiles in Mr. Simpson's *Select Exercises*" as follows:

"... as to those productions that bear the names of Waddington, Glennie, &c., they are so contemptible as to entitle their authors to no other part of the public indulgence but '*the running banquet of two beadles.*'" The italics of the quotation are by Burrows.

THE AXIOMS OF GEOMETRY.*

By H. G. Forder, B.A.

§1. It is generally recognised that the proofs given in the school text-books on Geometry are incomplete and that many theorems are assumed because they are intuitively 'obvious,' and one of my aims this afternoon is to show wherein this incompleteness lies and to state the 'obvious' assumptions.

Let us consider the standard proof of the theorem that "the angle at the centre of a circle is twice the angle at the circumference subtended by the

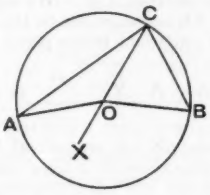


Fig. 1.

same arc." There are several cases to be shown, but for simplicity I take only the case in the figure, where $A\hat{C}B$ is an acute angle and the centre O is inside it. In the course of the usual proof the following theorems are assumed:

(1) The exterior angle of a triangle is the sum of the interior opposite angles.

(2) Angles at the base of an isosceles triangle are congruent.

(3) If α, β, γ, δ be angles (or rather the measures of angles) and $\alpha=2\beta$, $\gamma=2\delta$, then $\alpha+\gamma=2(\beta+\delta)$.

(4) If the ray† CX is inside $A\hat{C}B$, then the ray OX is inside $A\hat{O}B$.

Of these, (1) needs the Euclidean parallel axiom, (2) depends on the theory of congruence, (3) on the theory of magnitudes, (4) on the theory of order. As a rule, (3) is almost ignored and (4) is completely ignored in elementary teaching, though it is clear that they are both essential to a sound proof. (I do not suggest that they should be treated in *elementary* teaching.)

§2. *Congruence.* As I intend to work backwards to the foundations, I will begin with the theory of congruence. What do we mean when we say that two figures are congruent? We mean that the figures can be put into one-one correspondence so that if P, Q be points of one figure and P', Q' the corresponding points of the other figure, then the point couple (P, Q) is *congruent to* the point couple (P', Q').

It will be noticed that in defining 'congruent' I have used the word 'congruent' itself; but if the definition is examined it will be seen that what we have done is to define congruent *figures* by means of the simpler notion of congruent *point couples*. This is the sort of definition that is usual in mathematics. The term to be defined is defined by means of terms already introduced; these latter are defined by means of earlier terms, and so on; it is clear that this process must end sometime, that is, we must start with terms that are not defined. Call these terms 'undefined entities.' In the same way each proposition is proved from preceding propositions, these latter from earlier ones, and so on; the propositions from which we start we call 'axioms.' Thus at the basis of any deductive doctrine we have undefined entities and axioms, which are unproved propositions about these entities.

* A lecture delivered at the Annual Meeting of the Mathematical Association, Jan. 7, 1929.

† A ray is the set of points on a line on one side of a fixed point, the '*origin*' of the ray. Cf. below, § 7.

§ 3. Can we define the relation of congruence in the proposition " the point couple (P, Q) is congruent to the point couple (P', Q') " ? We shall at present take this relation as undefined ; later we shall see in what sense it can be defined. (We can *interpret* it as meaning that a pair of dividers whose points are at P and Q can be transferred so that their points take positions P', Q'. This interpretation is, of course, never used in the subsequent reasoning.)

Starting then with the notion of congruent point couples our definition of congruent figures tells us that the interval AB of a line is congruent to the interval $A'B'$, if we can set up a one-one correspondence between the intervals so that any point couple (X, Y) in interval AB is congruent to the corresponding point couple (X', Y') in interval $A'B'$. We naturally desire the theorem to hold that " the interval AB is congruent to the interval $A'B'$ if and only if the point couple (A, B) is congruent to the point couple (A', B')."

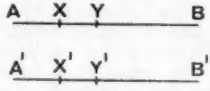


FIG. 2.

To secure this and other desirable propositions we introduce the axioms :

C. I. If (A, B) is any (ordered) couple of points, then on any ray $A'X$ there is just one point B' such that (A, B) is congruent to the (ordered) couple (A', B').

C. II. If (A, B) is congruent to (C, D), and (C, D) is congruent to (E, F), then (A, B) is congruent to (E, F).

C. III. If B is between A and C, and B' is between A' and C', and (A, B) is congruent to (A', B'), and (B, C) congruent to (B', C'), then (A, C) is congruent to (A', C').

C. IV. (A, B) is congruent to (B, A).

In discussing congruences we assume everything is known about the relations of order, *e.g.* the properties of ' between,' or what is meant when we say a ray lies ' in an angle.' When we say (A, B) is an ordered couple we mean it is to be distinguished from (B, A). In all the above discussions it was assumed implicitly that the couples were ordered couples.

The axioms we have given for congruence enunciate properties that would be easily overlooked or assumed without notice, and this is also the case for most of the other axioms I shall give. But the real proposition that any axiomatic treatment seeks to prove is that the axioms given are sufficient to build up the whole theory desired. The axioms and the earlier propositions may seem trivial, but the fact that they are a sufficient basis for the superstructure is not trivial ; and the problem of finding the smallest possible basis is often extremely difficult.

§ 4. Turning now to the congruence of angles (I note in passing that by an ' angle ' I mean simply two rays with a common origin, together with that origin) ; our general definition says that $A\hat{B}C$ is congruent to $A'\hat{B}'C'$ if the points on the angles can be put into one-one correspondence, so that if P, Q

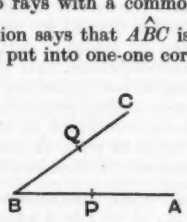


FIG. 3.

be points on $\hat{ABC}$ and P', Q' be corresponding points on $\hat{A'B'C'}$ then the couple (P, Q) is congruent to the couple (P', Q'), or, as we may now say, so that the intervals PQ and $P'Q'$ are congruent. We naturally wish that in this correspondence B should correspond to B' and not to some (other) point on the rays $B'A'$ or $B'C'$. It is not necessary to put this restriction in our definition if we assume the following axiom:

C. V. If ABC and $A'B'C'$ be triangles and D, D' are on the prolongations of the intervals BC, $B'C'$ respectively, and if AB, BC, CA, BD be congruent respectively to $A'B'$, $B'C'$, $C'A'$, $B'D'$, then AD is congruent to $A'D'$.

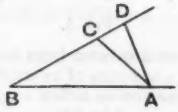


FIG. 4.

It can now be shown that in the correspondence of the angles above, B does correspond to B'; and further we can show, and so need not assume, that three non-collinear points cannot be congruent to three collinear points; in fact, that if B is on the interval AC and the figure made up of three points, A, B, C is congruent to the figure A', B', C', then B' is on the interval $A'C'$. But the range of theorems which can be shown from C. I.-V. in the presence of the order axioms, given later, is really amazing. It results from the work of R. L. Moore and Dorroh that they suffice to show all the ordinary theorems on the congruence of triangles, also the existence of perpendiculars, and the theory of congruence in three dimensions. Dorroh has recently communicated to me a proof that these axioms suffice to show that all right angles, even in different planes, are congruent. This unexpected result shows that discoveries are still to be made even in this most ancient of all mathematical theories.

§ 5. *Magnitudes.* In particular the congruence axioms enable us to show that angles can be treated in many respects as a class of magnitudes. As an illustration of the abstract logical treatment, let us consider this point.

A class is called a class of magnitudes if it contains at least two elements and admits an operation, called addition, between elements, which from any ordered couple of elements a, b of the class produces a definite third element of the class, denoted by $a+b$; where the operation is assumed to satisfy the following conditions,* the elements indicated being in the class:

(1) $a+b$ is distinct from a. (2) $a+b=b+a$. (3) $(a+b)+c=a+(b+c)$. (4) If $a \neq b$, then either there is an element x such that $a+x=b$ or there is an element y such that $a=b+y$. (5) If a is an element, then there are elements x, y such that $x+y=a$.

Now what about the addition of angles? We should certainly wish the following proposition to hold. 'If a, b are congruent angles and x, y are congruent angles, then $a+x$ has the same measure, or magnitude, as $b+y$.' This suggests that we should put all congruent angles into one class and say that angles of that class 'have the same measure,' we can then speak of adding the measures of angles instead of adding angles. How we shall best introduce the new term 'measure of an angle' is a difficult question of logic with which there is not time to deal. If we suppose we know what is meant by the measure of an angle, it is an easy matter to define the sum of the measures of two angles. Denoting the measure of $\hat{ABC}$ by $\mu\hat{ABC}$, we say:

$$\mu\hat{AOB}+\mu\hat{CPD}=\mu\hat{HXK},$$

* These conditions do not provide for a zero magnitude; and, as a rule, we do not need a zero angle; in fact such an angle would be a nuisance.

if there is a ray XL in $H\hat{X}K$ such that $H\hat{X}L$ is congruent to $A\hat{O}B$ and $L\hat{X}K$ is congruent to $C\hat{P}D$; we then find that all the conditions in the definition

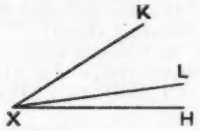


FIG. 5.

of magnitudes follow for these measures from axioms C, except that the sum of two measures need not be a measure of an angle; because, speaking loosely, we have not ascribed any measure to reflex angles, or even introduced them, as yet. We should not remove this difficulty simply by introducing reflex angles, because, speaking loosely again, the sum of two reflex angles need not be a reflex angle. The only solution is to bring in trigonometric angles of all sizes. This can be done on the basis of axioms C. The theory of the measures of intervals (lengths) is much simpler and follows from axioms C. I.-IV.

§ 6. But axioms C do not give all we want. By their means, although we can make a triangle elsewhere congruent to a *given* triangle, we cannot perform the following constructions:

'Three intervals are given, the sum of the measures of any two being greater than that of the third; to draw a triangle whose sides are congruent to these intervals.' To show that such a triangle exists we have to assume an axiom on the intersection of circles. It suffices to assume the following: 'If a circle K passes through two points collinear with the centre of another circle K', one point being inside, one outside K', then K and K' meet.' This axiom is assumed implicitly by Euclid in the proof of his first proposition. From this and the congruence axioms it can now be shown that if one circle has a point anywhere inside and another point anywhere outside a second circle, the circles meet in just two points, and that a line through a point inside a circle meets it in just two points. The whole theory of congruence now lies at our feet.

§ 7. *Order.* So far we have assumed that the order relations are fully known, inasmuch as we have used such terms as 'between,' 'interval,' 'ray,' 'inside.' I now proceed to this most abstract part of the subject, the discussion of the order relations.

When we say 'the point B is between the points A and C' we are affirming a three-termed relation between A, B and C. Other examples of three-termed relations are:

(1) For points A, B, C; $A\hat{B}C$ is a right angle.
(2) For numbers or vectors A, B, C; $A+B=C$.
(3) For numbers or vectors A, B, C; $A-B=C$.
(4) For people A, B, C; A prefers B to C.

Suppose we denote any of these relations by $[ABC]$, so that in the first case $[ABC]$ means '$A\hat{B}C$ is a right angle.' Then in the first case $[ABC]$ implies $[CBA]$ (since if $A\hat{B}C$ is a right angle, so is $C\hat{B}A$); in the second case $[ABC]$ implies $[BAC]$ (since if $A+B=C$, then $B+A=C$); in the third case $[ABC]$ implies $[ACB]$, and in the fourth case it implies that $[ACB]$ is false. These are formal properties of our three-termed relations.

Our problem is to find such formal properties of the relation 'between' from which all other properties can be deduced. The problem is here com-

plicated by the fact that we desire our geometry to be at least three-dimensional. The solution we outline is due to Veblen. We take 'point' as undefined, and we take an undefined three-termed relation between points. We affirm that this relation holds between points A, B, C by writing $[ABC]$. We assume the following axioms of order:

O. I. $[ABC]$ implies that A, B, C are distinct.

O. II. $[ABC]$ implies that $[BCA]$ is false.

Def. If A, B are distinct, the set of points, consisting of A, B and all points X such that $[XAB]$ or $[AXB)$ or $[ABX]$ or $[XBA]$ or $[BXA]$ or $[BAX]$, is called the '*line*' AB.

O. III. If C, D be distinct points of the line AB, then A is a point of the line CD.

O. IV. If A, B be distinct points there is at least one point C such that $[ABC]$.

O. V. There exist three points not on the same line.

It would take us too long to discuss these axioms with any completeness, and I will only point out that the only things involved in them besides the undefined 'point' and '$[ABC]$' are logical terms such as 'implies,' 'distinct,' 'false,' 'there is,' 'the,' 'or,' 'such that.' These and other needed logical terms have been thoroughly discussed by Peano, Whitehead, Russell, Hilbert, Weyl and Brouwer.

The next axiom to be introduced is the most fundamental in all geometry:

O. VI. If A, B, C are distinct points, A not on the line BC, and if D, E be points such that $[BCD]$, $[CEA]$, then there is a point F on the line DE such that $[AFB]$.

If A, B, C are distinct points, not on the same line, we can now define the '*plane*' ABC as the set of all points on all lines joining two points of the sides

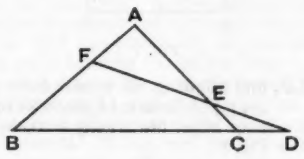


FIG. 6.

of triangle ABC, a triangle being defined as usual. It can be shown that between any two points lie an infinite number of points and that any three non-collinear points determine the plane on which they lie.

I turn to the definition of 'angle' which was needed above. If O, A be distinct points, the 'ray' OA is the set of points consisting of A and all points X such that $[OXA]$ or $[OAX]$. If OA, OB be rays not in the same line, then the '*angle*' AOB is the set of points on the rays OA and OB, together with O. Thus an angle is identified with what are usually called its sides. Another definition which at once suggests itself is to define an angle as the set of rays usually described as being contained within it, with or without its sides; but it is found that any such definitions lead to serious difficulties in the technical development.

So far we have stayed in the plane. To reach space we need only one other axiom:

O. VII. All points are not in the same plane.

If we wish to restrict ourselves to three dimensions, we need one further axiom which can take various forms, such as:

O. VIII. If two planes have a point common they have at least one other point common.

§ 8. *Areas and volumes. A digression.* From the planar axioms of order we can define a polygon, and its inside and outside; and show that a polygon separates its plane into these two regions; and that the inside of a polygon can be split up into triangular regions such that no two have a common inside point, and such that any point inside the polygon is inside or on a triangle, and that any point inside a triangle is inside the polygon. These theorems might perhaps be despised as obvious, but to deduce them from our axioms—which is the real question—is by no means easy. But the axiomatic method suggests questions of fact as well as of deduction, and the answers are often surprising and sometimes sensational. Thus when we split up our polygon into triangles we can secure, if we wish, that the vertices of all the triangles are at vertices of the polygon; but the corresponding theorem is not true for solids. In fact there are polyhedra which cannot be dissected into tetrahedra whose vertices are at vertices of the polyhedron. The simplest such polyhedron is the one recently discussed by Schönhardt (*Math. Ann.* 98, 1927). If we take a prism whose base is an equilateral triangle and whose side walls are squares, and, keeping the base fixed, we screw the top of the prism round through 30° keeping AA', BB', CC' of constant length, but lengthening the

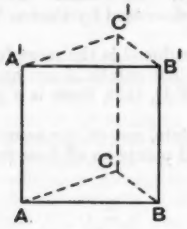


FIG. 7

diagonals AB', BC', CA', and allowing the square faces to cave in and crease along these diagonals, we get a polyhedron of the kind required, since the join of any two vertices, if not an edge, lies wholly outside the polyhedron. (A model was shown.) (See Fig. 8.)

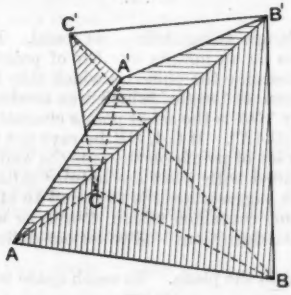


FIG. 8.

The existence of such solids is one reason, though not the most important, why the theory of volumes is so much more complicated than that of areas. Let us compare these theories: Given two polygons P, P' of equal area, can we divide P into triangles so that P' will result from P by mere rearrangement

of these triangles? The answer is, we cannot always do so on the basis of the axioms already assumed, and that is why in dealing with the fundamental theorem of the equivalence of parallelograms we must at one stage *subtract* triangles; but we can do so, if we assume an additional axiom, the axiom of Archimedes.* Turning to volumes, we ask: Given two polyhedra P, P' of equal volume, can we always split P into tetrahedra so that P' arises from P by mere rearrangement of these tetrahedra? The answer here is No, even if we assume the axiom of Archimedes; for it can be shown to be impossible to split up a regular tetrahedron into tetrahedra and fit the pieces together to form a cube; nor is it of any use here to allow subtraction as well as addition.

If, however, we assume a full continuity axiom,† such as that given below, and allow ourselves to split up the inside of a polyhedron into point-sets of any kind, scattered about anyhow, and not merely forming the insides of tetrahedra, then it follows that any polyhedron P can be split up into a finite number of point-sets which will fit together to make *any other* polyhedron of *any* (finite) volume, or for that matter to make any sphere, ellipsoid, and so on. Similar theorems hold on the surface of a sphere (Banach and Tarski, *Fund. Math.* vi.). I think these results can justly be called sensational. But in a plane, if we split up a polygon into a finite number of point-sets, and fit them together to make a new polygon, we can only get a polygon of equal area. We do not know, and I make this present to our circle squarers, if it is possible to split up a circle into a finite number of *point-sets* and fit them together to make a square.

§ 9. *The continuity and parallel axioms.* If in addition to our order axioms we introduce a suitable continuity axiom and a parallel axiom, we have all that is needed for constructing Euclidean geometry, for on this basis we can introduce congruence by definition. As a continuity axiom we may take:

Ax. K. *If* the points of an interval AB be divided into two sets such that (i) each set contains at least two points, (ii) the sets have no common points, (iii) no point of either set is between two points of the other set, *then* it follows that there is a point C in one of the two sets which is between every point ($\neq C$) of one set and every point ($\neq C$) of the other set.

This axiom corresponds to the axiom of the Dedekind section and need only be assumed for one interval. It can now be shown that parallel lines (*i.e.* coplanar, non-cutting lines) exist. Finally, to make our geometry Euclidean, we can assume that, given a line and a point not on it, there is not more than one parallel through the point to the line; though it is not necessary to assume so much as this.

§ 10. *Congruence by definition.* I have said that from our axioms of order, continuity and parallels, the whole Euclidean geometry can be erected. It is necessary to indicate very briefly how congruence may be brought in. It is clear that if each point of space were associated with three real numbers (xyz), which we may call its coordinates, then we could *define* the 'distance' of two points whose coordinates were (xyz) and $(x'y'z')$ as the positive square root of $(x-x')^2+(y-y')^2+(z-z')^2$, and say that the point-couple (P, Q) was congruent to the couple (P', Q') when the distance PQ equalled the distance $P'Q'$. We could then test our congruence axioms. Everything therefore hinges on assigning coordinates to points, and this depends in turn on placing scales along the axes. How are we to do this without using congruence? The germ of the idea of the following investigation is seen in Fig. 9. Take three non-collinear points A, B, X. Assign to A, B the numbers 0, 1 respectively. From these three points our whole figure can be constructed simply by joining points and drawing parallels, though it is clear that our figure

* See last year's Presidential Address.

† The results following assume the Multiplicative Axiom.

assumes certain propositions, of which later. To ... B', C, D ... we assign the numbers ... -1, 2, 3, ... Thus some points at all events on AB have

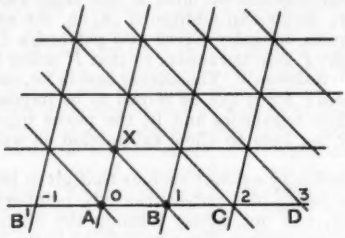


FIG. 9.

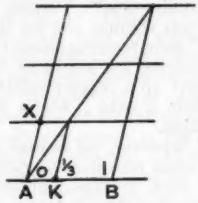
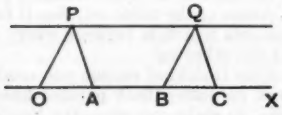


FIG. 10.

numbers attached to them. Fig. 10 shows how we can attach numbers to some points between A and B. If now we assume our continuity axiom, we can, by generalising this process and taking limits, make each point of the line AB correspond to a unique real number, and conversely; and in doing this we need only assume the axioms of order, continuity and parallels.

It is, however, more interesting to postpone the continuity axiom, but then it is not certain that our construction will enable us to reach every point of AB; (in fact, if our line AB is an ordinary continuous line, it *is* certain that we *cannot* in this way reach every point, but only the 'rational' points). So we proceed as follows: Take two points O, U on the line OX on which we wish to place a scale. If in Fig. 11 we have $PQ \parallel OA$ and $OP \parallel BQ$ and



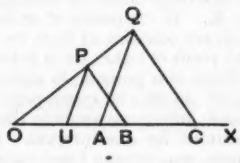


FIG. 11-12.

$AP \parallel QC$, we define the sum of points A, B as the point C, and write $A+B=C$. If in Fig. 12 (P, Q being any points collinear with O) we have $UP \parallel AQ$ and $BP \parallel CQ$, we define the product of the points A and B as the point C, and write $A.B=C$.

Now comes the main fact. The ordinary laws of algebra can be shown for these sums and products from the following special cases of Desargues' and Pappus' theorems.

Desargues' theorem. If ABC, XYZ be coplanar triangles, and the joins AX, BY, CZ be distinct and either parallel or concurrent, and if AB, BC be respectively parallel to XY, YZ, then $CA \parallel ZX$; conversely, if AB, BC, CA be respectively parallel to XY, YZ, ZX, then AX, BY, CZ are parallel or concurrent.

Pappus' theorem. If A, B, C be collinear and A', B', C', collinear, and AB, $A'B'$ be distinct lines and CB', CA' be respectively parallel to BC', AC', then $BA' \parallel AB'$.

It was these theorems that were assumed also in Fig. 9. Pappus' theorem is only needed to show $A.B=B.A$.

With this method the points themselves serve as coordinates, for they can be added and multiplied, and the laws of ordinary algebra are obeyed. Once

we have scales along the axes, it is not difficult to assign three coordinates to each point in space. But how shall we show Desargues' theorem and Pappus' theorem without using congruence? The first follows easily from our order axioms and the parallel axiom, provided we use all our order axioms and not merely those which refer to the plane. But Pappus' theorem does not follow from the axioms of order and parallels alone; it is necessary to add another axiom, and the continuity axiom is quite sufficient. But when we add this axiom we have achieved another end, for now the points of any one line can be brought into one-one correspondence with the real numbers in such a way that if, in that line, A, B correspond to the real numbers x, y, then $A+B$ corresponds to $x+y$ and $A.B$ to xy. Defining distance and congruence as above, we can now show our congruence axioms, and our axiom on the intersection of circles. The final result is that the axioms of order, parallels and continuity enable us to erect the theory of congruence and the whole of the Euclidean geometry. Every single step leading to this conclusion is the result of a series of researches beginning with the Greeks and continued until the present day.

§ 11. *A brief note on projective geometry.* This geometry knows nothing of order, congruence or parallels. Taking 'point' and certain classes of points called 'lines' as undefined, it can be based on the following axioms: (1) Two distinct points fix just one line on which both lie. (2) If A, B, C be distinct non-collinear points, a line which meets BC and CA in distinct points meets AB. If A, B, C be distinct non-collinear points, we can now define the 'plane' ABC as the set of all points on all lines joining A to points of BC; if A, B, C, D be distinct non-coplanar points, we can define the 'space' $ABCD$ as the set of all points on all lines joining A to points of the plane BCD. (3) There are at least three points on any line, there is at least one line, all points are not in the same plane, but all are in the same space.

From these axioms we can set up a scale on any line, very much as above, parallel lines being replaced by lines which meet on a given line fixed once and for all, and Desargues' and Pappus' theorem being replaced by their projective equivalents. Now the former can be shown from the above axioms, the latter cannot. It must be assumed as an extra axiom. All projective geometry then follows.

§ 12. *On making a geometry projective.* Let us now omit our parallel axiom, starting only with the axioms of order and continuity. Then, although it can be proved that at least one parallel can be drawn to a given line from a given point not on it, we do not know if just one can be drawn or an infinite number. We can, however, make our geometry projective, *i.e.* we can find combinations of figures in our geometry for which the axioms of projective geometry hold. Let a, b be two coplanar non-cutting lines. Let any two planes, one through a, one through b, meet in the line c; let any other two such planes meet in d. From our order axioms alone it can be shown that c, d are coplanar, and a really simple proof of this would be a step forward. Now consider the set of lines $[L]$ which are cuts of planes through a and planes through b, together with the set of lines which are cuts of the plane ab and planes through the lines $[L]$; any two lines so obtained are coplanar. We say the set defines, or rather *is*, a projective point. It replaces the point at infinity where these lines are vulgarly said to meet. We can also define projective line and projective plane; these things satisfy our projective axioms, including Pappus' theorem, but to show the latter we must assume a continuity axiom of some kind or other. Thus, *via* projective geometry, we can erect a co-ordinate geometry on the exiguous basis of the order axioms and the continuity axiom.

§ 13. *Two unsolved questions.* (1) To show that the medians of a triangle concur, from the axioms of order and congruence alone, without leaving the

plane and without using parallels (or of course proportion or areas). This deduction should be possible since the theorem is true in the classical non-Euclidean geometries. It has been shown with the above restrictions, except that space was used. (2) To find a deformable closed polyhedron, *i.e.* one which can be continuously deformed by altering its dihedral angles, but keeping its faces of the same size and shape. It is known that such a polyhedron cannot be convex; but so far as I know, no one has yet constructed a non-convex closed polyhedron, whose faces do not interpenetrate, which is deformable, nor has anyone shown that such polyhedra do not exist.

§ 14. *Logical questions.* Given a set of axioms, what sort of conclusions can be deduced from them? This question has been considered in recent years by Hilbert and his followers with reference in particular to logic and arithmetic. It has become increasingly clear that it is not possible to deduce any mathematical doctrine from logic alone. Our choice of axioms and definitions is of course guided by experience, but I am not referring to that fact but to the development of the doctrine. In any deduction recognisable symbols such as letters or mental images have to be used, and they must be recognised as the same in the various places where they occur. This becomes obvious when we reflect on the earlier proofs in the theory of order. The six orders in which *ABC* can be taken have to be distinguished, and I do not see how it can be recognised that there are just six orders except, in the end, by exhibiting them and counting, and this seems to involve something which cannot be expressed in terms of formal logic, and though we do not actually have to use the theorem that there are just six orders, yet this theorem must be in the background if the work is to be significant. It thus seems that properties of the symbols are being used which can be exhibited but which cannot or need not be *stated*. Thus it seems to me immaterial whether we use figures or not; but if we use figures we must read from them only those properties we have agreed to read from them; if we use symbols we must use only those properties of their arrangements we have agreed to use. And of course mathematical intuition is needed just as much when we employ rigorously logical methods as when we are careless. Logic restricts us to certain paths, but it does not propel us along them. Logic without intuition is blind, intuition without logic is empty.

The reduction of the axioms of geometry to those of order would seem final and has a philosophical significance. For a space does not differ from a line in the number of points it contains, but in their order. A set of points of space is a set of elements in certain order relations to each other. That the congruence relations are secondary is also clear, for in the space of ordered points many types of congruence are possible, that one which we have selected for daily use being suggested by the behaviour of solid bodies. The corresponding mathematical development is to deduce this theory of congruence from the previous theory of order, or to insert it into that theory. The straight line did not figure among our fundamental undefined entities, but it was actually excluded by a technical device. If we wish really to exclude it, *i.e.* if we do not wish to assume there are sets of points such that two points of one set determine it and so on, we should have to erect a theory of point-sets like that of Hansdorff in the first edition of his *Mengenlehre*. In this theory one of the oldest geometric problems, that of defining the dimension of a set of points, has recently been solved.

§ 15. *Questions of teaching.* I need hardly say I do not advocate that the strict logical treatment of geometry, involving as it does abstract considerations on order, should be presented to beginners: in the beginning we must sin against the light and be sternly illogical. When the demands of scientific rigour and of teaching conflict, the former must be ignored, but when they coincide, it is inexcusable to use illogical proofs. For instance, proofs by

superposition should be scrapped; beginners look on them with just suspicion, and in, *e.g.* the congruence which concerns two angles and a side, the proof is easier to write out if superposition is not used. Again it is inexcusable to define a tangent to a circle as a line which meets the circle in just one point, and then to assume that all its other points are outside the circle; or to show the converse theorems about angles in a segment in the way that is usual; or to define the radical axis of two circles as the locus of a point from which tangents to the circle are equal; or to assert that parallel lines meet at infinity, without very careful explanation of what this phrase means; or to ignore the *single* point at infinity in the theory of inversion in a plane.

Let me also plead for exact statement of theorems. We need not impose it on our pupils, but we should impose it on ourselves. How rarely, for instance, is the theorem of Desargues on perspective triangles properly stated; how often is the phrase 'in general' used to cover a multitude of omissions; how rarely is the field in which we work defined, *e.g.* if two conics through four points be rectangular hyperbolas all are, and yet through any four points (in general, of course) pass two parabolas; how rarely are exceptions noted, *e.g.* when we say the locus of centres of four-line conics is a line, which is by no means always the case.

One serious blemish occurs in the very beginning of geometry. All school texts I have seen use the word 'line' in two senses. The line AB may be the infinite line or the interval AB. This double use actually causes acute confusion in the minds of beginners, and the fact that it has endured so long should cover us with shame. If we puzzle our pupils by using this frequent word in two senses, what right have we to expect any exactness of expression from them. Permit me to suggest that 'line' be used only for the infinite line, and that the word 'interval' be used for the piece, and, further, that the sign AB be used, when alone, only for the former, and not for the latter.

I grant to the full that in its beginning geometry should be taught in the modern way, and that everything which a boy considers obvious should be assumed without more ado; but I think that by the time he has reached matriculation stage the rigour should have arrived at much the same standard as Euclid reached, avoiding, I do not say ignoring, incommensurables. This standard will not be attained by a mere uncritical selection of Euclid's own proofs. For those who intend to study mathematics at a university, more should be required after the matriculation stage, for if they proceed to Cambridge they will be plunged into the rigours of modern analysis of the real variable. This will not be so strange to them if they have previously treated the well-known elementary geometry in rigorous fashion at school; and if questions on these lines were set in the Entrance Scholarship Examinations, this treatment would soon become common. The work could be preceded by a sketch of the modern logic of Peano (that of Aristotle is as useless as the Ptolemaic astronomy) which is easily accessible in his *Formulario* and in Russell's books.

A century ago analysis yielded in rigour to geometry, the younger sister has since become more exacting than the older, it is now high time that the balance were restored and the same rigorous logic used in both domains.

I have been able to deal to-day with the fringe only of a very wide subject, a subject which is by no means yet exhausted, and which is fertile in new points of view, and the study of which is, I think, of special importance to those engaged in elementary teaching. H. G. FORDER.

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643. A STAGE ILLUSION. In an appeal yesterday for a return to "the good old days," Sir Gerald du Maurier . . . described how he started work as an office boy in the City with a salary of 8d. a day, which was exactly eight times more than he was earning at the moment, for he was now out of a job.—*Daily Mirror*, Jan. 26, 1929 (per Prof. E. H. Neville).

THE MATHEMATICAL ASSOCIATION.

TUESDAY MORNING, 8th January, 1929.

Miss E. R. Gwatkin, M.A., opened a discussion on the question: "Should a candidate for the School Certificate be allowed to take, in place of the Mathematics and Science Group, a Group containing Drawing and Music and possibly other subjects?"

Mr. President, Ladies and Gentlemen,—I understand that no vote is taken on the result of this discussion, but it is clear that I am in the position of the proposer of a motion, and I therefore state at the outset my conviction that a candidate for a School Certificate should be allowed to take in place of the Mathematics and Science Group, a group containing Drawing and Music and possibly some other subjects. The tail of the subject as stated sounds a little vague, and I do not propose to devote much attention to it. The reason for its presence is that in the School Certificate regulations laid down by certain examining bodies the Group containing Drawing and Music also contains a miscellaneous collection of other subjects. I should be prepared to include Domestic subjects, and some sort of manual work which might be attractive to boys, but not such subjects as Shorthand or Bookkeeping. I should, however, be well satisfied if my hearers would agree with me in accepting either Drawing or Music as a substitute for a Group III subject, and my arguments will be directed to this end.

The subject as put before you is a narrowing down of a wider one. Seven English universities or groups of universities—Oxford, Cambridge, Oxford and Cambridge Joint Board, London, Northern Universities Joint Board, Durham, Bristol—hold examinations on which the School Certificate is awarded, and the Central Welsh Board has a similar examination. The examination is supposed to be suitable for all boys and girls between sixteen and seventeen who are up to the average—or, rather, it is intended for all except those who are very much below the average in intelligence—and have received a satisfactory secondary education up to this point. In all cases also it is possible by doing sufficiently well in a certain selection of subjects to obtain matriculation or entrance to the university or universities conducting the examination, and, under certain conditions, to all the others. We are however concerned, not with matriculation qualifications, but with the School Certificate alone, not with those boys and girls who intend to proceed to a university, but with the great m[illegible]ss of those who do not. It is of course true that many, probably most, of th[illegible] who obtain matriculation do not actually enter a university. It may well be that some of these also might be better advised to spend what is for many of them their last year at school on a slightly different collection of subjects than those required for matriculation at present. To that extent we are concerned with these candidates also.

As at present constituted all School Certificate examinations contain four groups whose contents are indicated on your paper. At present a candidate must satisfy the examiners in three groups, and in the first three, the fourth is in a subordinate position. My real contention is that the fourth group is as important in itself and as valuable for boys and girls who are attracted to it as any of the other three. It will probably be recognised by all as undesirable that qualification in four groups should be compulsory. It follows, therefore, that choice must be made, and I contend that choice should be free. Personally I should accept any three, throwing the English group into the melting pot with the others. But as far as I know this view has not been put forward by any official body of supporters, and I will therefore restrict myself to that which is pressed by the Association of Head Mistresses and endorsed by 94 per cent. of their members, that Group I be left sacrosanct, but that a candidate be required to qualify in any two of the remaining three Groups

II, III, IV. This is of course equivalent to saying that he may take Group IV instead of Group II or Group III, but to my mind the latter phraseology has a wrong emphasis.

What we plead for is a recognition of the value of aesthetic subjects in the education of the young, without any depreciation of subjects which, in this country at least—it was not so in the ancient world—have come earlier to recognition.

The change proposed is really a subject for discussion by a joint session of the Classical and Modern Language Associations as far as freedom to drop Group II is concerned, or by a joint session of Mathematical and Science Associations if freedom to drop Group III be considered. It is in truth a question of general education, for all educationalists to consider, and a question of general value for all who are interested in values as well as facts to consider.

It is, however, no doubt well for an association representing a particular branch of education to consider a general question. It is not an ingrained habit of specialists so to do, and it is obvious that they will wish to consider the effect of any change upon their own particular subject.

Probably some of you will expect me, a renegade mathematician in your eyes, to adopt an apologetic tone, but I do not feel called upon to do so. Those of my own generation will know that my mathematics was originally of the pedestrian order, I never marched with discoverers; moreover, in the life of the head of a school there is little chance for the slowest of walks. But I do care greatly for my subject, and I enjoy teaching it both to the more and the less intelligent as I enjoy few of my other present occupations. I would not have thought that my present views are the result of a middle-aged forgetfulness of the joys of mathematics. But I believe also in the value of other subjects for which I personally have less taste, and I make no apology for so doing.

For one action of mine I should appear in a white sheet. I mention it first, in honesty I hope, also, I confess, lest it be cast in my teeth later. In what seems now the dim distant past early in the war years I signed the report of the Government Committee on the teaching of Science, in which appeared a recommendation that a pass in Science should be compulsory in School Certificate. It would be easy to say that I had changed my mind, and I should be in no way ashamed of a new conviction born of experience. In fact, I am ashamed of my signature. I did not feel as strongly then as now, but even then I did not really want Science (or probably anything else) compulsory on *all* candidates. The trouble was that I was the only woman on an influential committee powerfully chaired by Sir J. J. Thomson; any reservation on this point by me would have been universally regarded as a demand for a soft option for girls. At that time I thought it better to let the girls go in with the boys, and hoped that in due time greater freedom would come for all. Now I do not care, the time is long overdue, and I only regret that I had not then the courage of my convictions.

One word with regard to matriculation. I should support the same view of the value of the four groups, provided that any university faculty reserved the right to make special requirements, *e.g.* a pass in Latin for an Arts degree, or in Mathematics for a Science degree. There is something also to be said for a pass in a modern language for scientists, but I cannot see that the future Classical or History specialist is more benefited by an elementary knowledge of Physics or Botany than by the study of Art or Music.

Now we return to the School Certificate. With the present grading the boy or girl who aims only at the School Certificate, or who may aim at matriculation but only gets School Certificate, is either weaker all round than his fellows, or uneven in power, possibly very good on his own line but with a weak point. But my line of attack on the present system is not that a girl (and from now on I will ask you to allow girl to stand for boy or girl unless

I say otherwise), a girl who cannot do Mathematics or Science, should be allowed to take Drawing or Music instead, but that a girl who can do Drawing or Music best should be allowed to devote a considerable amount of time to that subject to the exclusion of others for which she has less interest or aptitude. It is true that she can at present take Drawing or Music as one of her five subjects. But the fact remains that if either of these is her good subject, she has to pass in three other groups as well, one of which may easily be distasteful or weak, and require an inordinate amount of time, whereas if her best subject is in one of the other three groups, there are only *two* others in which she must qualify. The artistic girl is handicapped compared with her sister of more academic tastes.

If we consider the history of education in English boys' schools, we see that the content of the curriculum has increased and increased. New subjects have been added without any corresponding ousting of the old ones, though some ousting there has been. Originally, mainly Latin and Greek with a modicum of Ancient History and a little Mathematics. Then more Mathematics, a little French, a little English, and later a little Modern History. Then more Mathematics still, more History, and more—though mostly very bad—French. Then Science took the field with a rush; Classics declined and schools arose which knew no Greek, and where even Latin was at a discount. German followed in the wake of Science, and French was more seriously taught. Drawing, first the handmaid of Science, gradually acquired for its own sake a minor place in the curriculum. Music, very good in some schools, in others negligible. The course was rather different in girls' schools which came into being in the middle of the process and which already had a heritage of attention to the arts. Domestic subjects have in many cases been added, and in boys' schools manual work. In practically all schools to-day there is what one must call a very crowded curriculum if all branches (even if not all subjects) are studied by all pupils, or a very varied curriculum if selections are made.

All would agree in theory that a general education should include English subjects, foreign languages, or at the very least *a* foreign language, Mathematics, some Science, some Drawing, some Music, though the two latter may receive only lip service. With regard to Drawing I should say that Art is a more descriptive term perhaps; most examining bodies make the History of Art a choice among the subjects (usually three in number) which may be offered. Music does not entail performance upon an instrument, but there is an aural test as well as a paper on theory.

Take the subjects:

GROUP I. English, History Ancient and Modern, Geography:
probably studied by all pupils at some time in their school career.

GROUP II. Latin, Greek, French, German:
probably all make at least a beginning in two languages, and many take three.

GROUP III. Mathematics, Physics, Chemistry, Botany, Zoology (or combined Biology):
all take Mathematics and one or more Sciences.

GROUP IV. Art, Music:
taken by all girls certainly up to a point, and at any rate by many boys. Music may be represented only by Singing, but aural training is on the increase.

It is a truism that examination should follow curriculum. Some pupils, and I am afraid occasionally some teachers, behave as if pupils learn subjects in order to be examined on them and for that reason only. There is truth in the criticism brought against us by foreigners that we are examination mad.

Still every one would agree that it would be impossible that every candidate should be examined in all the subjects which should exist in the school curriculum unless the nature of the examination were considerably altered. And it would require alteration in two ways: first, syllabuses would have to be much reduced, a change in many cases immediately desirable, and also—by no means the same thing—the standard would have to be lowered. At present in all School Certificate examinations it is necessary to pass in five subjects. In some, Arithmetic is a compulsory sixth subject; in others, only if no other Mathematics is offered. It is arguable that a better test of the sum total of the school education would be afforded by a wider examination at a lower standard, but I should not myself desire it. One practical objection would be that probably such an examination could not be made suitable for matriculation, leading immediately to two groups of candidates and much difficulty about border-line cases. A sounder, because educational, argument is that it is not desirable that so many subjects should be studied in what is for many their last year at school, and that they should at least have an opportunity of doing work at a good standard in subjects for which they have a taste. It would matter less for obvious matriculation people; let them offer as many subjects as they can carry, and if the standard were lower their opportunity for specialized work at a high standard would come in the sixth. But it is just these General School people, whose education may be co-terminous with their school life unless the desire to continue it for themselves be implanted in them, who should not be driven by the necessity of offering a large number of subjects, and who should above all be encouraged to take any subject they care for. If they are good at anything, they should take that subject as a matter of course and the rest in descending order.

And here I would combat one fiction. Drawing and Music are *not* soft options. I believe a high percentage of candidates offering these subjects are successful, which may be misleading to the lay mind. The fact is, of course, that as a rule only candidates who are distinctly good at these subjects offer them. It would be just as useless to say that a girl who was weak at Mathematics should take Drawing instead, if she could not draw, as *vice versâ*, and even more absurd to say that a girl who had no language sense and could not do French should take Music instead if she had no ear, than the converse proposal.

If, then, the examination be kept roughly at its present standard there must be choice, and the question is on what principle. There is some sense in the present grouping, but to my mind Mathematics and Science are not as closely allied as is often made out. I understand that there is a certain type of scientific mind which is not specially mathematical, the biological mind so sadly uncatered for in boys' schools we learn. There are certainly mathematicians whose subordinate interests lie in Arts rather than in experimental science. I am one myself, and Dr. Whitehead used to give me a most comfortable feeling of being in good company with him in this respect. And as regards educational training there seems to me to be more in common between Mathematics and the linguistic side of Latin than between Mathematics and, say, Botany. I can understand the position of people who say that a candidate must take Mathematics, and the position of those who say that he must take Science, but I am doubtful about the equivalence of one for the other.

In any case for the type of pupil considered the curriculum is presumably much the same for all up to the end of the pre-certificate year. There may be a choice of languages, French and German, or French and Latin, or a choice of Sciences, Physics or Chemistry or Botany or two of the three. Mathematics for the majority, though a few may have dropped to Arithmetic only. Geography might be at a discount in some schools, but English and History would be continued, and at any rate in girls' schools Drawing and Singing and possibly some other form of Music, except for the entirely ungifted.

In the examination better pupils will take seven or eight subjects, the

majority at least six, but for some the minimum of five will be best. Hence there must be exclusion. My contention is that for some pupils a course of, say,

English, History, Geography, French, Music

would certainly be better than

English, History, Geography, French, Chemistry,

or a course

English, Geography, French, German, Drawing

more interesting than the same course with Mathematics substituted for Drawing. There may not be many such, but there will be some, and it is the interests of the minority that I plead. Our English boast is always that we give freedom. I spent a year in Germany once before the war (I believe things are altered now) and saw a number of schools. There the problem of choice among subjects had to be solved by the pupil at a very tender age, or presumably by his parents who would send him to a Gymnasium if they wished him to learn Latin and Greek, a Real Gymnasium if he was to learn Latin but no Greek, a Real Schule if he was to confine himself to Modern Languages, specialists in science would be found chiefly in technical schools, though not entered at so early an age. We have in most schools all subjects possible. We allow choice, but we penalize the artistic or musical girl or boy; his good subjects are extras, frills which may be added on to the more serious ones, but that is all.

And yet are these subjects of such minor importance when we consider later life? We have lived through the early age of the development of science; its discoveries still astonish us daily, and the powers which Mathematics has bestowed upon twentieth-century man, and the enlargement of vision resulting therefrom would have been beyond conception a hundred years ago. But there is also now a growing demand that things should be beautiful and offend neither eye nor ear. There was always a reaction against the crude manifestations of Science, its machine-made hideosities for instance, but it tended to be backward-looking, freakish, and contemned of the ordinary world. Now we recognize that machine-made articles have come to stay, but attention is being given to beauty as an accompaniment to efficiency. If we have lost beautiful horses from our street, some motor-cars (though not all) are beautiful in construction and movement. Some modern buildings are fine, though some are loathsome. There is much good music in the modern world, and we are actually at last awaking to the atrocities of the street noises which assail our ears day and night.

You may think I am talking at large and entirely off the point. It is true that these average boys and girls may not have much opportunity of creating beauty in their lives. But they do at least go to form public opinion, and if public opinion once seriously demands beauty, or even the absence of ugliness, it will in the end obtain it. If the real dignity of Art and of Music is fully recognized in the schools, something will be done for the future amenities of life.

It may be said that the importance of these subjects can be recognized in schools without their entering into the examination system at all, let alone their entering on equal terms with other subjects. Is it not a pity to bring them within its thrall? If the taking of an examination were at the will of the individual, well and good, but as things are examinations have to be taken, and if the time for serious study of a subject is to be found that subject must carry full weight in the examination.

Another objection may be made that if a subject ceases to be compulsory it will be less taught. I do not think this applies here, since at present Mathematics is not compulsory nor is Science. Some pupils may be deflected from the study of either, but in my opinion such pupils should be so deflected. And in itself I think the argument is a bad one. Good mathematical or

scientific brains are not easily turned from their path, and I do not think a subject suffers by having its tail of weak stragglers shortened.

Of course I think *all* pupils should learn Mathematics, and should continue to learn it (apart from practical necessities) until it is clear either that they are making no headway, and are not likely to make any more, or that they are much more interested in other things and their time can be better employed. I need not here attempt to define the value of the study of Mathematics, those who teach it know, the rest may blaspheme as they will. But there are pupils who are incapable of assimilating its virtue, others who reach the saturation point very early, others who get more from other subjects.

The same is true of the sciences, though there are probably few who are incapable of imbibing some knowledge of some science, whether infused by the true scientific spirit is another matter. Some study of natural phenomena is desirable for all, partly as knowledge and partly as appreciation of scientific methods of acquiring knowledge. Some experimental work in a laboratory all should carry out that they may have some understanding, however elementary, of what is meant by scientific research. But whether such study should cease, for those not specially scientific, when they are between fifteen and sixteen, or should be carried on for another year and examined, seems to me a matter for determination in the case of each individual.

My experience of course lies entirely with girls, and I expect to be told that girls cannot do Mathematics, and there is an end of it. Some cannot and some can. Even with boys some cannot and some can, I understand, though very likely the proportion is different. In any case those minorities are provided for under the present School Certificate regulations. My point is that the artistic and musical pupils are not provided for, or at any rate are at a considerable disadvantage.

Another argument which I know carries weight with many headmasters is that employers now recognise the School Certificate, and that they would no longer consider it of the same value if Group IV were placed on a level with II and III. This is the erection of a convenience into a fetish. We struggled to lessen the number of external examinations, entrances to universities and to various professions, and to get them standardised, and a very convenient result has been obtained. Still I do not think that, avoiding invidious distinctions, all employers would now consider the certificates of all examining bodies of exactly the same value, and I cannot think that any intelligent employer would consider all certificates obtainable under any one examining body of exactly equivalent value. The Groups passed in would be endorsed on all certificates, and he could take his choice. The validity of the argument depends upon the value attached to Group IV. If a certificate showing success in Groups I, II, III is satisfactory evidence of a general education, and a certificate showing success in Groups I, II, IV is not, then of course the employer is right in his view; if the value of Group IV is admitted, all that remains is that the employer should be taught the error of his ways as soon as possible; in the meantime it is open to him to confine his attention to certificates which do please him.

But it is time some effort was made to change public opinion, and it always means a definite effort to realize another point of view.

I was reading the other day a small book by an Oxford philosopher called *The Construction of the World in terms of Fact and Value*. The author says of pure science: "in the last resort nothing counts with the exception of facts and the interconnections of facts. A fact may be beautiful or ugly, attractive or repulsive. All that science asks is whether it exists." "In the world of pure science we reach a region of pure fact, in which value as such has been entirely eliminated." On the other hand, "in art we may be said to attain a world of pure value. For it is a world dependent throughout upon value for its very existence." And the writer adds: "Both these worlds are only

very partially accessible and but dimly revealed to the majority of mankind. And those who live and move freely in the one are likely for that very reason to find a greater difficulty in approaching the other."

As it happens Mathematics and Music are not an uncommon combination, and the mathematician who is also a musician should be able to appreciate the point of view of the musician who is no mathematician. Of the scientist I have less hope, but I am addressing mathematicians and will leave it at that.

Mr. **B. L. Gimson** (Bedales School) confessed that his early training had been chiefly in engineering and in science, so that in speaking for the other side he might perhaps seem to be disloyal to those early days. Nevertheless, he felt that more adequate attention and true value should be given to the aesthetic side of education. There were boys and girls who were weak at mathematics and who could find their real outlet in aesthetic studies. Surely such pupils should be given credit for that. It was not merely a question of alteration of the curriculum but of giving those gifts full value, full credit by examination. He had just spent an interesting term in America visiting schools of a progressive type attempting experiments in curriculum-making. He had visited twenty-five or thirty schools, though not all were progressive, because he wanted to get a perspective view. However, he had been much impressed by the remarkable work done in art and crafts subjects in at least twenty-five schools attempting progressive work. Those subjects were not regarded as soft options but as a real form of expression for pupils unable to express themselves as adequately in the mathematical studies. Unfortunately, one noticed there—as he supposed was true of schools in this country—that those progressive tendencies seemed to grow less as pupils approached the examination age. He felt, however, that the English School Certificate Examination had far more progressive tendencies than that in America. But if America was attempting progressive work in art and music, why should not English teachers and examinations keep pace and give full credit by allowing Group IV to take equal prominence with the other three Groups.

He read a sentence or two from the evidence given by those called to give opinions to the Consultative Committee on the differentiation of the curriculum for boys and girls in secondary schools, because it emphasised the importance of the question for boys as well as for girls. The report said: "In boys' schools, with some notable exceptions, the aesthetic side has hitherto been much neglected. Several of our most authoritative witnesses, basing their opinions partly on the results of psychological research and partly on experience gained in co-educational schools, thought that the response of the two sexes in music and art was probably equal, if equal opportunities were provided, and that in consequence a more serious development of aesthetic training was very necessary in the whole of our system of secondary education." Thus it was not girls only who would benefit by the change which he felt sure was coming.

Again, in giving prominence to art and music, were mathematics and science to be given up altogether? Surely not. Children interested in the art side wanted a type of mathematics not quite of the kind they had to take at present in the School Certificate Examination, much as one might admire the type of papers set there. Much of that was of a rather formal character. Surely it was possible to give a kind of mathematics which would interest and be useful to that type of pupil; which would enable them not to drop out of their experience that kind of training but which would enable them to develop more fully an aesthetic type of education. He was thinking particularly of the kind of mathematics that dealt with design; the mathematics needed by architects. The mathematical idea approached from the arts point of view was what should be aimed at when teaching mathematics to the aesthetically minded type of pupil. He supported, with the utmost conviction, the proposition on the agenda.

Mr. **W. Hope-Jones** (Eton), after consultation with his drawing and musical colleagues, had come to the conclusion that the danger of soft options was negligible because the standard both in drawing and music was high: there was no real danger of a pupil taking up those subjects as an escape from mathematics. If with music one taught something of theory and harmony, and with drawing something of the value of perspective, it seemed to him that pupils would learn something of mathematical principles, if not the name of mathematics. Therefore he was in favour of allowing drawing and music as alternatives, but not to the science and mathematics Group only. At present there were three Groups and one below those on a lower scale of sanctity. He was not in favour of taking one of the three holy ones and making it unholy by putting it with the group below, because that was a degradation. He was, however, in favour of promotion by lifting up the inferior subject, putting it on a level with the rest, so that there would be no distinction between the value of one Group and another, but that a certificate could be given on three out of four Groups.

Mr. **F. H. Knight** (Friends' Central Education Committee, Friends' House, Euston Road, N.W. 1) said his whole bias was in favour of blue-sky mathematics, and that he spoke for boys only, knowing nothing about girls—at least he had never taught any though many had taught him. So he was not looking for soft options for girls. Next, he was out rather more than those who had spoken for the elevation of handicrafts to at least the level of art and music, though he enormously valued those two subjects.

He was thinking particularly of two boys he had taught: one, curiously enough, was born in Cambridge, but was always bottom of the secondary school. Though a mathematician, he had been entrusted with bottom French and had spent the whole of one term trying to teach that boy French. At the end of the term the boy knew "Je suis"; at the end of the holidays he had forgotten it. At one period during the holidays he had stayed at the boy's house in Cambridge and discovered that though the boy could not do a single bit of what was usually described as brain work his bedroom had been lit by acetylene gas generated by plant the boy had made. He admitted he did not sleep all night for fear of being blown up, knowing the boy! But, seriously, the boy had a great aptitude for that kind of thing, and had built the plant as a hobby in his spare time out of pure interest. He had not been taught anything about it in school, where he could not rise above the bottom place. That boy was also a brilliant football player and one of the finest examples of physique he had ever seen. Yet when he applied for the navy the boy was accepted on intellectual grounds and rejected on physique—it was really only a matter of colour-blindness. At any rate, the navy may have been justified there, but how they accepted the boy on intellectual standard he had never been able to fathom.

The other boy he had in mind had also remained at the bottom of the school up to leaving time, but after the war had become head of a big reconstruction column in France, and made a name for himself by putting up houses and reconstructing devastated areas, about which he had learnt in his spare time in workshops at a boarding school run by Quakers, who, as a matter of fact, had rather a reputation for encouraging hobbies of all kinds. Boys were trained through touch and other senses as well as through the mind. He contended that boys and girls should be able to take a certificate which would assure possible employers that they had had a good education—an education not secondary in the limited sense in which the word was generally used, but more like the education given in a good modern or central school in London or any other leading place.

He was claiming that the mind was developed by far more attention to the senses, to touch, sight, hearing and so on, and that the School Certificate examination should not be set by universities, though the latter had a perfect

right to set Matriculation. In spite of the fact that it had been said that Matriculation should be left out of the discussion, it came in in that if a child had chosen his subjects for the School Certificate, say, in the northern universities, he could get on with credit in handicrafts. But if he had a chance of taking the Matriculation he could only get it with credit in Groups I, II or III; a credit in handicrafts was no good there. In that particular respect there was difficulty, because a child did not know what to do. Again, a child educated properly through his senses would take even the present Matriculation in his stride, and his capacity for mathematics, languages and so on would be stimulated by his handicraft or other work. But boys being human and the subjects demanded in the School Certificate being what they were, both masters and boys would tend to work to the examination syllabus and to neglect the handicraft side of their education.

In regard to what the last speaker had said, Mr. Knight added that he was inclined to put English in the holy of holies, though not an English teacher. That inclination arose from the lack in the world of the things of the spirit. He would certainly put the other Groups on a level. It could not be said that all was well with education when admirals, generals, politicians, archbishops and others said on speech days that they had always been bottom at school! He did not believe there was anything like a near approach to the way to educate the human mind, otherwise boys at the bottom of the school would not become archbishops and so on, and those at the top would not afterwards get lost in life. That side of the problem was so familiar that there was a tendency to smile at it, but it really did constitute a criticism of education as at present constituted.

Finally—and he stressed this most emphatically—what characterised civilisation at present was that science had found its place. Mathematics and science had come into their own in recent times as compared with classical training. Although as a mathematician he should not say so, scientific discovery had carried the development of material things far beyond the development of the human spirit. That was a very serious feature in present-day life. If materialism got on much faster and spirit lagged behind much more, civilisation would blow up as that acetylene gas plant might have done. Teachers who had the future in their hands had to see that the spirit caught up. Surely art and music would train the things of the spirit; handicrafts also would enlarge the human spirit by giving it a number of different points of contact with life in general.

The **President** said there was one point which had not been brought out as much as he had expected. It was said that the proposed change would lead to deterioration in mathematical teaching in schools. He did not think Miss Gwatkin had dealt with that point.

Professor **Steggall** (St. Andrews University) felt that the Association existed for the improvement or advancement of mathematics, and regarded as one of the evils of the present time competition among universities in connection with obtaining students. One of the results of that had been a tendency to put in option after option. As a matter of fact, he held the somewhat old-fashioned idea that in a university there were certain fundamentals which should be insisted upon. He had very little sympathy with many of the modern university degrees; there were other institutions which could give degrees in art, engineering, and so. Among the fundamentals for ordinary intelligent people who claimed to enter a university some knowledge of number and form was essential. A person who was a real intellectual freak and came forward for a degree as understood by old-fashioned people should be told that the universities did not provide for every kind of freak. If they endeavoured to do so, the speaker was convinced universities were making a very great mistake. He did not wish to hurt Miss Gwatkin, but he sometimes trembled when he had listened through somebody else's wireless at a musical

performance and heard the most excruciating howls from saxophones. In conclusion, the speaker said he would be sorry if the Mathematical Association gave unqualified approval of a proposal which meant the discouragement of one of the fundamental elements in the proper education of the intelligent human mind—namely, the elements of number and form.

Miss **C. Waters** (Somerset) pointed out, as an examiner of some years' standing, that the gain would be enormous if a vast amount of energy on the part of teachers were not expended in a forlorn hope of trying to push unsuitable pupils through the London Matriculation.

Mr. **W. C. Fletcher** (Highgate) thought, in the first place, that Miss Gwatkin's very orderly and fair statement of the matter could be summed up by saying: the question that Miss Gwatkin ought to be discussing is, What is the proper curriculum? not the question she professed to be discussing, What is the proper examination? Miss Gwatkin had said that it was a mere truism that examination ought to follow curriculum, but she tossed it aside as if it was a truism that did not matter, whereas it mattered very much indeed. The head mistresses, if he might say so with respect, were fundamentally wrong at present in attacking examination when they ought to be attacking curriculum. So long as they did that those to whom they brought those attacks would hold the head mistresses suspect, as they had been held hitherto, of seeking for soft options.

Secondly, Miss Gwatkin set out very fairly that there were two kinds of girls: the gifted girl who was going to be good at drawing or music, and the weak girl—as the weak boy—who had difficulty in getting on at all. The problem of the gifted girl was one thing; that of the weakling another. The misfortune was that even Miss Gwatkin in her orderly statement—and still more those who spoke, as he was doing, on the spur of the moment—hopelessly confused those two cases. Until they were carefully disentangled and one knew precisely which was being dealt with, arguments would again be suspect and treated as of little account.

Thirdly, the question was not one as to the value of art and music. It was quite ridiculous to suppose that those who, years ago, drew up the examination regulations were contemptuous either of the one or the other. That was an irrelevant point to bring into the discussion. While relevant when considering what the normal curriculum ought to be, it was not reasonable to assume that because in the examination drawing and music were not on the same footing as the other subjects, that therefore they were regarded as of no account.

The speaker's fourth point was practically the first over again. What lay behind the original suggestions for the examination requirements was the then generally accepted idea of the normal minimum curriculum. There was no question, of course, as regards schools on the grant list; they had to make provision for English subjects, for at least a modern language, for mathematics and science. Miss Gwatkin was perfectly right: if logic had been followed there ought to have been four groups, mathematics one by itself, and science by itself. That was deliberately avoided so as not to make the strain too great. Anyhow, all those subjects had to be represented in the curriculum, and substantially represented. That was common agreement at the time the regulations were drawn up, and the examination, quite rightly and properly, followed it. If it was desired to upset that state of things, the proper step was to upset the underlying theory, and make it clear that for the normal boy and girl it was not right to require that they should pursue some element in each of the three groups right through the first part of their school life. That might be true; he could not say whether it was or not. All he was saying was that at the time the examination requirements were set up it was the accepted doctrine, and until that doctrine was upset the difficulty had not been faced. Until they had established the fact that it was right and

proper for pupils to drop the whole Group—that was what it meant—they had not established their case.

Again, the question of school organization and discipline was involved, for that was wrapped up with the curriculum. Was it to be a curriculum for everybody or one with which it was possible to play fast and loose? His memory went back further than that of many present, and he distinctly remembered the sort of work that the Board's officers had to do in their early days. They had to fight head mistresses on the point as to how much licence—not freedom—they should take or give to individual girls. In some schools long lists were submitted of girls for whom it was desired that mathematics, or other subjects, should be dropped. That sort of thing was pretty well dead, but it appeared there was a desire to recall it. That was another reason why the proposals were disliked. Were those concerned going to stick to a fairly uniform curriculum or merely going to follow individual whims? There the question between the gifted and ordinary girl became of vital importance, because nine out of ten at least of the requests for exemption did not come from gifted girls but from those who were lazy; whose previous education had been bad, or who were being very badly taught at that moment. Surely it was not pretended that schools were perfect, or anything like—it was known they were not.

Mr. **J. Katz** (Selhurst Grammar School) had expected to be in the position, which he very much enjoyed, of being the one person in favour of the resolution. As theirs was rather a conservative Association perhaps the majority of members were of the opinion that nothing would be done for quite a long time so that the resolution did not really matter, or perhaps those present were so chivalrous that they would not like to oppose the forlorn hope led by Miss Gwatkin. Or was it that the members of the Association were afraid of committing professional suicide if music and drawing and other such subjects were substituted for mathematics—a subject sometimes regarded as disagreeable. But was it a fact that mathematics was unpopular? He had no doubt that in some schools and with some pupils it was still, perhaps, the most unpopular subject, but in most schools to-day—certainly in the majority of boys' schools—mathematics was an exceedingly popular subject. He would go so far as to make the claim that in an increasing number of schools and with an increasingly large number of pupils mathematics was popular because it was one of the most stimulating subjects on the curriculum.

He agreed with Mr. Fletcher that the deeper questions involved were not those of examinations but of curricula. It so happened that at the last meeting of the London Branch he had read a paper entitled "On what grounds can we defend the traditional position of Mathematics in the curriculum?"

He took it that mathematics differed from most other subjects in the curriculum in this respect: that mathematics—school mathematics at any rate—was the introduction of the child to the art of thinking. He admitted that other subjects involved a great deal of thinking, but in mathematics there was a very large amount of thinking applied to a very small amount of empirical material, whereas in other subjects there was, on the whole, a relatively small amount of thinking applied to a large amount of empirical material. Mathematics was such a valuable subject in the curriculum because it concentrated a large amount of thinking on a very simplified and small amount of empirical material.

It seemed to him that the key to our present-day difficulties lay in the fact that there were a certain number of children in the schools so made mentally that they could not get this invaluable training in the art of thinking from mathematics, but could get it from other subjects, such as history, geography, manual work. He thought he was right in saying that very little was at present known as to how many such pupils there were. Were those pupils to be compelled to get their mental training through mathematics?

He would go so far as to say that every teacher of almost every subject was really a teacher of mathematics, in the sense that every teacher was really a teacher of the art of thought. When history and geography were taught on modern lines by really competent teachers few would challenge the statement that those subjects provided an extraordinarily valuable training in the art of thought. Could it really be said that in mathematics the pupil obtained a training in a very distinctive kind of thinking which mathematics, and only mathematics, could supply? If that was so, then teachers must insist on mathematics in the curriculum for every child. But did they really hold that view? One could name a large number of famous men—Goethe, Coleridge and others—who had had a very poor opinion of mathematics because it contributed nothing to their mental training. Perhaps they had been taught by less competent teachers than those present, but the fact remained that there were types of mind which were not stimulated by mathematics. Teachers of mathematics had to accept that position, and they might do so without any kind of fear that their bread and butter would be interfered with.

As to the School Leaving examination, he would go so far as to say that if a child reached a really high standard in geography, English, history, art and music—he was not sure about making French compulsory—and had at the same time kept up the arithmetic of the junior primary school, then that child was entitled to be considered a reasonably well educated young person. That being the case the speaker suggested to the members of the Association that they should not insist on mathematics for everybody. There was no fear whatever that their great subject, which had held a high place in the curriculum for many centuries, would lose that place.

Miss **Gwatkin** in reply said: There are only two points I want to answer: one, Mr. Fletcher's astonishing assertion that I mentioned as a truism that examination should follow curriculum, but then tossed it aside. That was the very last thing I intended to do. My contention is that having art and music already in the curriculum we are warranted in attacking the examination. The last thing I should wish to do is to toss aside the idea that examination should follow curriculum. Those subjects are in the curriculum—possibly more in girls' schools—and therefore ought to be in the examination on the same terms. Then Mr. Fletcher said that at the time the School Certificate Regulations were formed the reason for the absence of Group IV. was the general idea of the normal curriculum for schools. . . .

Mr. **Fletcher**: I did not refer to the absence of Group IV. It was not absent.

Miss **Gwatkin**: I am sorry—I did not mean to misrepresent—the form which the School Certificate took was due to the normal idea of the curriculum at that time. My point is that that general idea has changed, and that the examination must change with it. Mr. Fletcher and those who agree with him may think the time is not ripe: the time never will be ripe unless those who believe put forward the necessity for the change. My contention is that the time is over-ripe.

The only other point I want to make is that it is easy to talk about playing fast and loose, but if you call playing fast and loose with the curriculum what other people call variation, then you beg the question. If you call licence what we call liberty, again you beg the question. If you call a boy or girl weak who wishes to study one subject rather than another, again you beg the question. Some children have whims, of course; not all. Mr. Fletcher spoke of fighting the head mistresses about the licence given to individuals; I think the head mistresses felt that they had better means of judging whether a particular choice made by an individual was one which would lead to a lifelong taste and should be accepted, or whether it should be rejected as really a whim.

The President referred to the point raised in connection with the deterioration of mathematical teaching, and here I should like to say that I think I am in my right position in the Mathematical Association. I care enormously for my subject. I know very little about art and music, but I think it is just for those who care about their own subject to try to appreciate the point of view of those to whom it does not appeal. It is the most difficult thing in the world to see another's point of view, but if the teachers cannot do it, it is hard on the pupils coming under their charge. In regard to depreciation of mathematical teaching, Miss Waters had really answered the point. Further, I cannot see that it is going to spoil the teaching of mathematics or the value of it, if those who either cannot do it—not therefore weaklings all round—or who dislike the subject, are removed. The remainder will be much more easy to teach and will not suffer by being dragged down by the tail.

After a short interval the chair was taken by Mr. ALFRED LODGE, who called upon Mr. Chignell to read his paper on the next subject.

THE USE AND ABUSE OF FORMULAE.

BY N. J. CHIGNELL, M.A.

When I first undertook to read this paper, I felt that I ought perhaps to apologise for introducing such a very elementary subject, because all of us are familiar with the use of formulae—some of us are only too painfully familiar with their abuse—and there is not much that may be said about these useful labour-saving devices that is not obvious; yet it seems to me that it is just in these very elementary matters that much harm is done to mathematics as a branch of education by faulty ideas and faulty teaching, and I want, if I can, to put before you as clearly as possible what I consider should be the purpose of formulae in education.

I am thinking of mathematics as a real means of education; that is to say, as a training which helps in all the varied problems of life and not solely in those problems that concern the engineer, the physicist, or the actuary.

For formulae play a very large part in the life of the modern world; indeed, they have played a part in the life of the world since the earliest times, and an appreciation of our attitude towards formulae at the present day can only, I think, be arrived at through the history of formulae in the past. In religion, in ethics, in law, in that most mysterious thing, social law, even thirty years ago, the lives of men and women were largely governed by formulae which had been dictated to them by their parents or their clergy, or their doctors or lawyers, and very seldom was it thought either necessary or desirable to question these formulae. Even now, wherever the standard of education is low, superstition and blind obedience to unexplained formulae enter into the lives of men and women.

To me the most hopeful sign of the present time is the fact that men and women are demanding something more than this.

It is my object, then, in this paper to try to express ways in which this new attitude to life and its formulae may be assisted by the teaching of the class-room; an ambitious aim, but one which, I think, is well worth the attempt.

A formula may arise as the result of a generalisation of a common experience, or it may have been arrived at empirically.

There are two distinct ideas underlying formulae and their use: (*a*) mere substitution of numbers for symbols, (*b*) the construction of formulae.

There is a process intermediate between these two that is often called "changing the subject of a formula," which involves the manipulation of the symbols of a given formula without actually making a numerical substitution, and without necessarily understanding what the symbols represent.

For full appreciation, all these processes require the power of translating verbal instructions into algebraic formulae, and *vice versa*.

For, after all, a formula is only a short instruction telling the user exactly what he has to do to reach certain conclusions from certain data. To illustrate my point, and to bring out, if I can, the real difficulty of the process, I give three simple examples.

$$\text{(i)}\quad s=(u+v)t/2.$$

Here is the "English" translation of this formula: "In certain circumstances,* the distance described can be found by halving the sum of the initial and final velocities, and multiplying the result by the time." I am not for the moment concerned as to whether it is correct or not to speak of multiplying a velocity by a time.

$$\text{(ii)}\quad A=\pi r^2.$$

To get the area of a circle, square the radius and multiply by a number which happens to be approximately 3·142.

$$\text{(iii)}\quad \Delta=\sqrt{s(s-a)(s-b)(s-c)} \text{ when } s=\tfrac{1}{2}(a+b+c).$$

To find the area of a triangle when the three sides are known. Halve the sum of the three sides, and from this number subtract, in succession, each of the three sides; find the product of these three numbers and the first; finally take the square root of the result.

For beginners a most complicated story, tedious to listen to, and very hard to follow. And yet the algebraical presentation of it is quite beautifully simple and orderly. But it is impossible to get away from the fact that this formula is, after all, nothing more than a concise method of giving just those instructions which I have so heavily expressed in what I am pleased to call "English."

It is impossible to appreciate formulae of any description until one acquires this power of reading the instructions hidden in this shorthand of symbols. It is not absolutely necessary to be able to translate the formula into English. The best French scholar is not the boy who can produce the most perfect translations of French into English, but the one who can read French and arrive at the meaning underlying the words without necessarily being able to put that meaning into his own language. But early language lessons must inevitably be devoted to the practice of translation. In the same way early lessons in the use of formulae should be given to the translation of the formulae into something that is very nearly English. In particular, in dealing with quite young boys I should never, in talking, read the formula in what I might call "formulese," but should always read it in the form of an instruction.

Judging from text-books, and from the attitude of mind of boys, formulae seem to be taught as though the one and only object were substitution, and the real understanding of the language seems to be left entirely in the background.

I take an illustration from a text-book.

"The weight in pounds of a foot of iron piping is given by

$$W=\frac{3\cdot142tk}{144}(b+t)$$

where the thickness is t'', the bore is b'', and k lbs. is the weight of a cubic foot of iron. A pipe whose bore is 3″ and thickness $\frac{3}{8}''$ weighs 13·3 lbs. per foot. Find the weight of a cubic foot of iron."

This is not a bad example as such. It has the element of surprise so dear to the heart of the schoolmaster, viz. the possible confusion between the "weight of a foot of pipe" and the weight of a cubic foot of iron. But the

* And that, by the way, is one of the real troubles about formulae—that there is always a condition attached, a condition which it is practically impossible to incorporate in the formula itself.

point that I want to emphasise is that there is in this question, to the average boy or girl, nothing more than "identify the various symbols, replace them by the numbers, and then solve the equation you get." A valuable drill—I am not questioning the value—but it provides less than a half of the training which formulae should rightly give. Several text-books suggest that there is something more in it than this, but in the mind of the boy or girl "getting the answer" predominates. Even when a question is set on the construction of a formula, there is always the chance of getting the right answer by a judicious blending of the letters involved without really troubling about the processes.

I propose two principles which, in my opinion, should govern all this teaching.

(1) That the distinction between mere substitution and the construction of formulae be very clearly preserved.

(2) That a formula should very rarely be memorised or used for any purpose other than mere practice in calculation unless it has been led up to in such a way that the pupil could, if necessary, construct the formula himself.

The formula quoted above clearly belongs to the first type.

As an illustration of a formula of the second type we have the "Sine formula" in trigonometry—often used by those who do not know what a sine is.

There are certain formulae which at one stage may possibly be "given" and at a later stage derived, *e.g.* the volume of a sphere, but no boy ought to be expected to remember a formula which he has not at some time or other proved. It is, for example, in my opinion, bad teaching to give to any boy not conversant with the calculus a question involving the volume of a sphere and not to tell him the formula.

And if such a question is given, it must be recognised that it is merely for the sake of practice in calculation and not that the boy may in future be able to find the volumes of spheres.

There are, of course, exceptions. Nowadays so much work is done in elementary mensuration with circles and cylinders that it often happens that a boy does remember these formulae from frequent reiteration.

But my point still holds that a boy should not be required to memorise these formulae, and therefore I welcome the fact that such formulae are printed in the standard tables that are used in the School Certificate examination, even while I deplore the existence of the type of diary that a boy may buy for half-a-crown containing all formulae that are ever used in any school mathematics.

When formulae are used merely for the sake of practice in calculation, the only condition I would make would be the one referred to already, viz. that no calculations should be made until the formula has been carefully read and the instructions contained therein mastered. When a formula is used for any other purpose, in fact, for the practical purpose of finding a velocity, or the sum of an A.P., or a side of a triangle, then I would say emphatically that a formula should never be used by a boy unless and until he has had considerable training in the method underlying the formula; in fact, he should have had so much training that he could, if necessary, recapture the formula when it has passed from his memory.

I want to take several instances where, in my opinion, formulae are habitually misused.

(1) *The formula* $I=\frac{PTR}{100}$.

This formula always makes me shudder; for either the boy knows the principle on which it is based, in which case there is no need for the formula; or else he does not know the principle, and is therefore merely using an apparently arbitrary formula for producing a numerical result which he hopes will agree with the answer in the book.

(2) *The formula for solving quadratics.* I say definitely that no boy, except a mathematical specialist, should ever be taught or allowed to use this formula.

If he has been taught the method that produces the formula, he should always find the method itself quicker than the use of the formula for numerical solutions, especially in these days when square and square root tables are usually available, and the formula will be of real use to him only when he begins to learn something about the theory of quadratics.

(3) *A.P. and G.P.* I wish to go into this in detail, because it illustrates more clearly than anything else the way in which formulae should be introduced. I do use these formulae myself, but I seldom allow boys to use them until the very end of their work in this section.

It is a real education to start with a series.

$$S=7+10+13+\ldots+124 \text{ (40th term)}$$

(to discover, by a perfectly fair generalisation, that just as the third term is got by adding twice three to the first, so the fortieth term is got by adding 39 threes and is therefore 124).

To rewrite it
$$S=124+121+118+\ldots+7,$$
$$2S=131\times 40,$$
$$S=2620.$$

And then again
$$S=7+10+13+\ldots+(3n+4) \text{ (check)}$$
$$S=(3n+4)+\ldots 7,$$
$$2S=n(3n+11);$$

hence
$$S=\frac{n}{2}(3n+11) \text{ with abundant checks.}$$

After a time the boy finds that the amount of writing that he has to do grows less and less, until he is almost automatically thinking of the nth term as $a+(n-1)d$, and of the sum as $\frac{n}{2}(a+l)$, and when that time comes he is able to use these formulae in their real sense and not in any way as mere substitution practice.

(4) *Uniform acceleration formulae in dynamics.* Instead of getting on to the formulae in the first instance, the principle of the $v-t$ graph is established, and all problems are solved by direct reference to the graph or to a sketch of it. There are, of course, occasional difficulties, but it is just as easy to see that the figure requires three data for complete solution as to see that a set of five formulae can provide the remaining two unknowns when three of the numbers concerned are given.

It may be objected that the figure itself has the nature of a formula, and this is in a measure true; but it is very much nearer to the principle connecting space with velocity and time than to a collection of memorised formula, and it does, to some extent, guard against the old trap of assuming that all motion is governed by these formulae. I should like to go into this case in greater detail, but will only add my own experience in teaching the beginnings of dynamics to a division of not particularly intelligent boys.

I made them work through a complete set of examples in uniform acceleration before such a thing as a formula was suggested, each example being worked straight from the figure; I then introduced the formulae, and later gave them free choice to use the formulae or the figure. About half of them chose to use the formulae, and half stuck to the figure, but almost without exception, whenever a real difficulty arose (*i.e.* the answer they got did not agree with that in the book—for a boy will always get an answer of some sort), they all went back to the figure to rescue them from the difficulty. And that seems to me to be the essence of what we want to get at—that a formula may be a convenient aid, but that it must be very firmly based on a principle, and that the principle is always there in the background when the formula either escapes our memory or fails us for some other reason.

I do not claim that any of this is particularly original; I have merely tried to put into words what is really at the back of the minds of all of us. A few of the things I have said may be controversial, and may perhaps lead to an expression of contrary opinions. But I should like to conclude with an idea which was certainly new to me when I came across it a few years ago. It represents an attempt to get at the real meaning of formulae through the process of transformation—the process which in an earlier section of my paper I called the process intermediate between mere calculation and construction of formulae.

Take $A=\pi R^2$. Translate into English.

"Start with the radius, square it, multiply by π, and you have the area. Now reverse the sentence. Start with the area, divide by π, find the square root, and you have the radius."

Translate into shorthand and you get at once the formula

$$R=\sqrt{\frac{A}{\pi}}.$$

The boy can look on the thing in the form of a railway with various stations *en route*, where each step is reversed in passing through a station in the reverse direction.

Apply the method to the route from circumference to area (non-stop run through radius).

Area ← multiply by π ← square ← radius → multiply by 2π → circumference.

Carrying out the non-stop run we get at once the two formulae:

$$A=\pi\left(\frac{C}{2\pi}\right)^2, \quad C=2\pi\sqrt{\frac{A}{\pi}},$$

and these can be simplified if required.

Take a more complicated example (taken, I think, from a School Certificate paper):

$$P=\frac{9900T}{3D}\left(5-\frac{L+12}{60T}\right).$$

The problem is to find L in terms of the other symbols.

The route from L to P is: add 12 ... divide by $60T$ ... multiply by -1, ... add 5 ... multiply by $9900T$ ... divide by $3D$, and you arrive at P. Now reverse the processes, and you get at once

$$L=-60T\left(\frac{3DP}{9900T}-5\right)-12.$$

All this looks rather pretty, but the worst of it is that it needs a mathematician to see when the process is applicable and when it is not.

For example, it is not easy to apply it to this formula when the problem is "find T," so that I should hesitate to give this as a universal method for transforming formulae, but if used judiciously this very worst is also a "best," for it does lead even a stupid boy to see that a formula is, after all, only an instruction to carry out various operations, and success depends entirely on a clear understanding of these instructions.

With this in view I would frequently make boys paraphrase simple formulae, *e.g.* $y=3x^2+5$ would be read:

"square x, multiply by 3, add 5, and you get y."

The reversal of the processes gives

$$x=\sqrt{\frac{y-5}{3}}.$$

This is better than the appeal to magic—"change side, change sign."

Such practice in reading through several expressions of a similar nature

would probably insure against many of the elementary mistakes made in algebra, such as confusion between

$$3xy^2, \quad 3(xy)^2, \quad (3xy)^2.$$

or

$$4x^2+9, \quad 4(x^2+9), \quad 4(x+3)^2.$$

In the last three cases the reversal of the processes leading from x to y gives

$$x=\sqrt{\frac{y-9}{4}}, \quad x=\sqrt{\frac{y}{4}-9}, \quad x=\sqrt{\frac{y}{4}}-3.$$

May I conclude, as I started, by dwelling for a moment on this general question of principle and formula.

Most people drive cars nowadays, and many drivers have difficulty with their gears. Try to teach a novice (I speak feelingly, for I am a novice myself) the art of "changing down" and "double declutching."

Punch some months ago had an article on the subject, and the humour of the article lay entirely in the fact that it was a perfectly straightforward statement of the operation involved.

It is almost impossible to teach along these lines—mere memory will always play tricks and spoil the whole operation; but once get the principle of the relative speed of rotation of two shafts firmly established, and the novice will almost invent for himself the complicated formula. It comes slowly—tediously—through much careful thought and accompanied by many blunders; but in the end it becomes automatic, and processes that are automatic are very closely akin to formulae.

But the automatic can only be based on a principle, and in all the problems of life as in all problems of mathematics, success can only come from the firm foundation of a principle which is always at hand when some new circumstance arises that seems to invalidate the automatic process.

The **Chairman**, in opening the discussion, thought the point that a mathematical expression should be representative as well as written was most important. There was, however, one point he thought Mr. Chignell might have laid more stress on, namely, the formula advocated and used in some schools to allow of students avoiding the solving of quadratic equations: they used a formula which, to his mind, was not solving the equation but a substitute for the solution.

Prof. **Neville** endeavoured to reassure the few teachers—if there were any—who were a little afraid that to deprive them of too much use of formula meant the taking away of the only pair of crutches with which the less able-bodied of their mathematical students could hope to scrape through the various qualifying examinations. Those crutches were really of extremely little value. With regard to quadratic equations, for instance, the practice of setting the question in the form "without using a formula . . ." was quite common. That being so, he thought the matter could be left to the examiner, who always had it in his own hands. In his experience the percentage of students who were wrong when they put in the formula was extremely high; reliance on formula was really most dangerous for a young student. As to the use of formulae with regard to arithmetical progressions, year after year he had seen questions on arithmetical progression suggested at large public examinations and simply dropped out of the paper completely, because the answers had become mere quotations of formulae. The effect upon the examination of the habits of the teacher to teach formulae was simply that the question was dropped out of the paper altogether, or the examiners said, "This question although last in the paper is not worth quite as much as the first," and so the crutches, so to speak, were valueless when it really came to the point. The answers that the examiner got soon showed whether there was anything in the prevalent teaching of a particular piece of the subject, or whether it was mere 'formulization,' if he could use that word.

Miss **M. Punnett** understood Mr. Chignell to say that he would not teach, or at any rate consistently use, the formula for the volume of the sphere with boys and girls who could not establish it; that was to say, until they had reached the calculus. In a case like that would Mr. Chignell admit a practical establishment of the formula? For instance, if boys could establish by experiment (*e.g.* with home-made models and sand) the relation between the volume of the sphere and that of the cylinder, so that knowing the formula for the volume of the cylinder they could easily deduce that for the volume of the sphere, would Mr. Chignell accept that as a justifiable establishment of the formula rather than cut it out? It seemed a pity that boys and girls who would never go as far as the calculus should not have an opportunity of exercises in and acquaintance with interesting formulae which occurred in ordinary everyday life.

Mr. **J. Katz** thought it was necessary to be continually on one's guard against the misuse of formulae, though he agreed with what Miss Punnett and Professor Neville had said, that there was a considerable place for formulae provided they were used intelligently. One special use had not been adequately dealt with by the reader of the paper: and that was the economic value of formulae. After working dozens of quadratics, it was illuminating to the child if he was asked: Can the process be generalized? Very often the class made suggestions as to how that could be done. He thought the mathematical training involved in obtaining the s formula for the triangle was considerable, and the fact that a boy might use that formula mechanically was not so serious a problem or so difficult as had been suggested. To be able to use $I = \frac{PTR}{100}$, provided the boys had obtained this generalisation themselves, was evidence of mathematical insight. In the proper desire to maximise the boy's thinking power and prevent him from slipping into merely mechanical methods there was a danger of going in the opposite direction and of so disgusting the boy that he might begin to regard mathematics as a fussy and useless subject.

Mr. **N. J. Chignell**, in replying, said: May I express my gratitude to Professor Neville for his remarks on the formulae of progressions. I think we have realised that the arithmetical and geometrical progressions have been only illustrations in substitution, and I think we have tried to avoid them, but I believe they might be of value in the way in which I outlined them. Miss Punnett asked about the use of the volume of the sphere formula. I agree with her that quite young boys and girls should be allowed to use that formula for certain purposes, and I think it is justifiable if it can be established on an experimental basis. What I am really asking is that it should be recognised that if it is not based on some theoretical or experimental preliminary work, if it is merely, so to speak, a formula from the clouds, then it should be understood that the use of that formula is a use for calculation purposes only and nothing else. But I still agree that once it is established in some form or other, either experimentally or theoretically, that it is quite justifiable to use it freely.

With Mr. Katz' criticism I am really in entire agreement. I almost crowed when I saw him do the first formula because it was exactly what I should do myself; viz. going through the process sufficiently often to have the process in your mind and to be able to recapture the formula *via* the process, if it has ever escaped you. I made the point that formula should not be used, merely because I think that the process is easier than the formula. I agree that with some people the reverse might be the case. I admit that I left out the very valuable economic side. I did so rather deliberately, but there certainly is a very valuable economic side in formula. I very nearly got to that when I spoke about processes becoming automatic. It seems to me that when you reach a formula you are getting a considerable saving of energy by the process becoming automatic, and I have no objection, of course, to

people using and doing things automatically, but I am very keen and very urgent that they should not be allowed to use an automatic process until they have arrived at it by a long training in the things leading up to it. Whether that is necessarily theoretical training I should not like to say. I think very often it is generalization on purely empirical lines. I think the use of all formulae wants rather careful watching, and I think it should be used only after what I might call long training in the processes leading up to it. I still squirm about that formula:

$$I=\frac{PTR}{100}.$$

I see Mr. Katz' point, but at the same time I would regard a formula of that description merely as shorthand for a principle. It seems to me that shorthand is not necessary in this case. In the case of formula for the area of a triangle, shorthand is vital. In a case where it is nearly as easy to write longhand it seems to me the use of shorthand is a little bit dangerous. I can, however, see Mr. Katz' point that the reversal of that formula is a valuable process, but that is the process I have been trying to illustrate.

The **Chairman**, in proposing a vote of thanks to Mr. Chignell, said the paper could be summed up in the remark, that formula, if they are used, must be used intelligently and, if possible, they should be dispensed with, the method being used instead. He thought everyone present would agree with that and thank Mr. Chignell for bringing the point before the meeting. (Applause.)

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923. [R. 1. b.] *Some Kinematical Properties of a Moving Lamina.*

The following elegant result is well known and was set in one of the college examinations at Cambridge.

A lamina moves in any manner in its plane. Prove that the locus at any instant of points which are at inflexions of their paths is a circle which touches the loci in the lamina and in space of the centre of instantaneous rotation.

Whilst trying to prove this, the following generalization occurred to me.

A lamina moves in any manner in its plane. At any instant let I denote the centre of instantaneous rotation, A the point which has no acceleration, and λ the angle between the directions of velocity and acceleration at some point P in the lamina. Then the locus of points P for which λ is a constant is a circle passing through I and A and cutting the common tangent at I to the body locus and space locus of the centre of instantaneous rotation at an angle λ. By varying λ we get a system of co-axal circles and that one for which $\lambda=0$ is the circle of inflexions mentioned above.

Also the locus of points P at which the ratio of the velocity to the acceleration is constant is a circle which for different values of the ratio is a member of a co-axal system having I and A as limiting points. C. Fox.

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644. Make the plan of your fireplace opening triangular, that is, running to a point, or nearly so, at the back. An angle of about 180 degrees (that is, a square placed on the angle) suits well, but the precise angle must be adapted to the width of the opening and the depth available.—Wood Fires. The Right Kind of Construction. *The Times*, January 10, 1929 [per Mr. F. Puryer-White].

645. The Egyptian papyrus, known as the "mathematical papyrus," in the collection at the Hermitage Palace at Leningrad, is now finally deciphered (says a Moscow Reuter message). It belongs to the eighteenth century B.C., and contains twenty-five problems on geometry and algebra. It appears that the mathematical methods applied were almost 1500 years ahead of Euclid's, corresponding absolutely to those used at the present time.—The Mathematical Papyrus. 1500 Years Ahead of Euclid. *Manchester Guardian*, October 4, 1928 [per Mr. H. J. Woodall].

INFINITY IN PLANE GEOMETRY.

By Prof. E. H. Neville, M.A.

It has been suggested that a sketch, limited to the real plane, of the theory of homogeneous coordinates and points at infinity which is developed for complex space in my *Prolegomena to Analytical Geometry*, would be of interest to teachers who without wanting to go deeply into the matter are humanly discontented with the crude treatment that makes of infinity "the place where things happen that don't".

At the cost of some repetition presently, let us plunge straight into the middle of the subject. Let A, B, C be any three points which are not collinear, and let any three numbers f, g, h none of which is zero be attached as loads to these points. Then let x, y, z be any three numbers which are not all zero. If O is any point in the plane, we can associate with O the vector

$$fx.OA+gy.OB+hz.OC,$$

which we will denote by $\mathbf{p}$. From two points O_1, O_2 we obtain two vectors $\mathbf{p}_1$, $\mathbf{p}_2$. Since $O_2A=O_2O_1+O_1A$, and so on, we have identically

$$\mathbf{p}_2=t.O_2O_1+\mathbf{p}_1, \text{ where } t=fx+gy+hz.$$

If $t\neq 0$, this identity gives $\mathbf{p}_2/t=O_2O_1+\mathbf{p}_1/t$, implying that the step from O_2 with vector $\mathbf{p}_2/t$ leads to the same point P as the step from O_1 with vector $\mathbf{p}_1/t$. Thus the step from O with vector $\mathbf{p}$ passes through this point P whatever the position of O in the plane. Moreover, to multiply x, y, z simultaneously by any number k is to multiply $\mathbf{p}$ by k, and is to produce a step from O which still passes through P. In fact, for given ratios of $x:y:z$, the different steps from O lie in one definite line, and the lines associated with different positions of O all pass through a point P which depends on the ratios but not on O; the ratios determine a *star* or bundle of lines, composed simply of all the lines through P.

If however $t=0$, the original identity takes the form $\mathbf{p}_2=\mathbf{p}_1$, implying that the step from O_2 with vector $\mathbf{p}_2$ is parallel to the step from O_1 with vector $\mathbf{p}_1$. For given ratios of $x:y:z$, the steps from O lie in a definite line, but the lines associated with different positions of O are all parallel; the ratios determine a bundle of lines, but this is a bundle of parallel lines, a bundle without a vertex, or, in conventional language, a bundle whose vertex is at infinity.

When $t\neq 0$, it does not matter whether we regard x, y, z as coordinates of the bundle or as coordinates of the point P; when $t=0$, there is no point, and the coordinates can only be coordinates of the bundle. Alternatively, we may give the name of "ideal point" to the bundle itself in every case. When the ideal point has a vertex, this actual point and the ideal point are equally specified by the set of coordinates x, y, z, and we can safely speak of "the point" to which the coordinates refer, leaving the context to elucidate the ambiguity. When the bundle consists of parallel lines, "the point" is necessarily ideal, and is only the bundle itself under another name, but the specification of the bundle by means of the coordinates is no less straightforward; we say that the ideal point is at infinity, or is inaccessible, but the lines of which the ideal point is composed are just ordinary lines.

The treatment so far assumes that A, B, C are ordinary points. To expose a relation between cartesian coordinates and homogeneous coordinates, we must begin further back. Suppose that we are given one fixed point Ω, a vector $\mathbf{r}$, and a number t. Then with an arbitrary point O let us associate the vector $\mathbf{r}+t.O\Omega$, denoted by $\mathbf{p}$. As before, if $\mathbf{p}_1$, $\mathbf{p}_2$ are the vectors associated with two points O_1, O_2, then $\mathbf{p}_2=t.O_2O_1+\mathbf{p}_1$, and by the argument already used, $\mathbf{r}$ and t specify a definite ideal point, which is at infinity if $t=0$. Instead of taking for A, B, C actual loaded points, we may now take ideal points, specified with reference to some origin Ω by vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and numbers

f, g, h. Then the place of the vector $f.OA$ is taken by the vector $\mathbf{a}+f.O\Omega$, and given three numbers x, y, z, we associate with an arbitrary point O the vector $x\mathbf{a}+y\mathbf{b}+z\mathbf{c}+(fx+gy+hz)\,O\Omega$, determining as before an ideal point which has x, y, z for coordinates; the coordinates kx, ky, kz associate the same line with O as the coordinates x, y, z, whatever the value of k, other than zero, and this is what we mean by calling the coordinates homogeneous.

If the ideal point specified with reference to Ω by $\mathbf{r}$ and t is accessible, its vertex P is the actual point such that the step ΩP has the vector $\mathbf{r}/t$; if the ideal point is at infinity, it is composed of those lines which are parallel to the vector $\mathbf{r}$. Let ΩX, ΩY be two lines through Ω, let $\mathbf{a}$, $\mathbf{b}$ be unit vectors in these lines, and let A, B be the points at infinity formed of the lines which are parallel to ΩX, ΩY; also let us take for C the ideal point which has Ω for vertex, and let $h=1$. Then the point whose coordinates are $x, y, 1$ is the point specified with reference to Ω by the vector $x\mathbf{a}+y\mathbf{b}$ and the number 1, and this is the accessible point whose vertex P is such that the vector ΩP is $x\mathbf{a}+y\mathbf{b}$; in other words, P is the point whose cartesian coordinates with respect to the axes ΩX, ΩY are x, y. Conversely, the actual point whose cartesian coordinates are x, y is the vertex of the ideal point of which $x, y, 1$ are homogeneous coordinates. If $z \neq 0$, the ideal point which has x, y, z for one set of coordinates has $x/z, y/z, 1$ for another set, and is the accessible point whose vertex has the cartesian coordinates $x/z, y/z$. But the ideal point which has $x, y, 0$ for a set of homogeneous coordinates has no actual point corresponding to it, and no corresponding pair of cartesian coordinates is to be expected.

Developments may be left to the reader. An ideal line may be defined in several ways, but in any case it is a class of ideal points whose coordinates satisfy a homogeneous linear equation. In general, if this equation is $lx+my+nz=0$, there are actual points whose coordinates satisfy the same equation; these actual points compose an actual line, and the ideal line consists in fact of those bundles to which this actual line belongs; the actual line is called the axis of the ideal line, and the ideal line is said to be accessible. But the equation $fx+gy+hz=0$ is not satisfied by the coordinates of any actual point; it is the equation satisfied by x, y, z if and only if the ideal point x, y, z is a bundle of parallel lines, and to say that the points at infinity lie on a line is only to say that by giving the name of "line at infinity" to the class whose members are these bundles, we avoid making a verbal exception of an equation which in itself is perfectly intelligible. E. H. N.

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646. Arithmetic in the seventeenth century.

Matthew Duke, at Masulipatam, to the East India Company, August 27, 1621. "The keeping of these accounts would not be very tedious yf they were ordred by denomenacions of moneyes that hath even parts in it, as 8 cash to the fanam and 16 fanams to a pagodo, which will produce no les fraction than ½ cash, but being kept at 9 cash to the fanam and 15 fanams to the pagodo breeds intollarable fractions not fitting merchants accounts, for there is more troble to reduce the fraction then about all the rest of thaccount; and when all is done, not the value of 3*d*. found more or les. But when these things shall com to the generall bookekeeper to enter, what a labrinth it will bring him into. But these things would be ordred at home by yourselves that for the future these erors may be avoyded (which I may justly cale erors); and yf I thought I should be tyed to theis intrecat accounts, I would rather serve other waise without wages then with doble wages continue in them, for although I know well enough how to reduce a fraction to his least denomenacion, yet to bee much troubled with them is tedeous, and rediculus to keep them in such great account."—*English Factories in India, 1618-1621*, ed. W. Foster. (Oxford, 1906), pp. 263-4 [per Mr. F. Puryer-White].

PARABOLIC MOTION OF THE PROJECTILE.

By E. J. Atkinson, M.Sc.

Every student of Mechanics or Applied Mathematics has at some time or other considered the motion of a particle projected into space with a known velocity. He is generally content to ignore air resistance and like contingencies, and to take the subsequent motion as that resulting from the combination of the constant velocity in the given direction of projection with the acceleration due to gravitational attraction upon the particle. Mathematical manipulation of the statements giving the position in space of the particle after an interval of time shows:

(*a*) That if the particle be projected into space with a given velocity the range on a given plane is a maximum if the direction of projection bisects the angle between the plane and the vertical. Further, the direction of motion of the particle when it strikes the plane at this point of maximum range is at right angles to the direction of projection;

(*b*) Also, if two directions of projection be obtained such that the range on the given plane for a certain velocity of projection is the same for each, then it can be derived that these directions make equal angles with the bisector of the angle between the plane and the vertical.

In order to demonstrate these phenomena practically, a means is necessary whereby a body may be made capable of a velocity in one direction and of an acceleration in another, the one motion being independent of the other both in magnitude and direction. Those members who were present at the Annual Meeting of the Association in January 1925 will remember that the writer exhibited an arrangement, subsequently called the "Vector Trolley Apparatus," which allowed of the combination of motions in two directions. Each motion was independent of the other and could be an acceleration or a velocity as desired. As then developed the propositions which could be demonstrated were the parallelograms of velocities and accelerations, the principles of relative motion, Newton's second law and the parabolic motion resulting from the combination of a velocity with an acceleration. A description of the

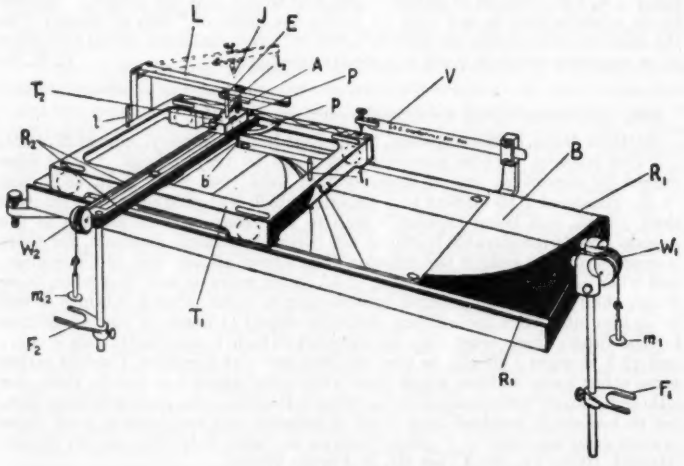


Fig. 1.

apparatus and of these demonstrations appeared in the *Journal of Scientific Instruments* in July and August 1925 (copies may be obtained on application from Messrs. G. Cussons, Ltd., Lower Broughton, Manchester).

The apparatus is shown diagrammatically in Fig. 1 below. T_1 is a frame-trolley which can move along the track R_1. It carries upon its back, as it were, the small trolley T_2 which can move along the track R_2, which track can be arranged normal or inclined to the path of the main trolley. This small trolley carries a marking brush b, which leaves an ink trace as it passes over a sheet of paper pinned to the base-board B.

Since each trolley is made to move forward by the action of a falling mass, the motion in each case when the mass falls freely is one of uniform acceleration. If the falling mass is arrested in its fall, as it may be by the arrestment forks F_1 and F_2, the ensuing motion will be one of uniform velocity. Each trolley, *i.e.* the main trolley and the small trolley, may be loaded and controlled separately. For demonstrations in which a velocity and an acceleration are combined, the frame-trolley is allowed to move forward with an acceleration in order to acquire a velocity with the cross-trolley held back by the guider or wire-arm A, the arrestment fork being so arranged to remove the actuating falling mass at the instant that the cross-trolley arrives at the point of release from the guider. From this point forward the cross-trolley will move over the paper with a motion which is the combination of a velocity with an acceleration. If the directions of the independent motions of velocity and acceleration are at right angles, we have a trace which may well represent the path of a bomb released from an aeroplane flying in space (air resistances, etc., neglected)—the bomb having as an initial velocity forward that of the aeroplane of which it has been a part, while gravitational attraction causes the downward acceleration, the direction of the velocity representing the direction of projection.

With a shell or cricket-ball projected into space, if the projection is at an angle θ, (say), with the horizontal, hence at $(90° - \theta)$, to the direction of the acceleration, again vertically downwards caused by gravitational attraction.

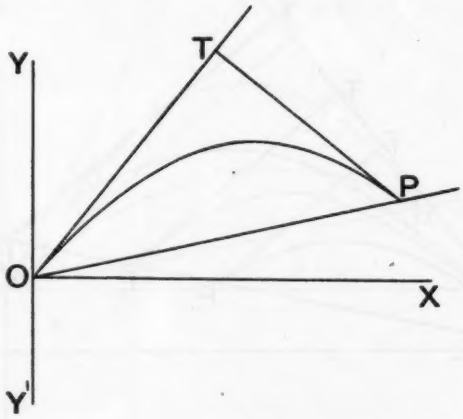


FIG. 2.

If the cross track is inclined so that it makes some angle greater than 90° with the main track, the excess will represent the angle of projection, the

small trolley moving with an acceleration when released will give the direction representing the vertical.

The trace obtained as a result of this combination is a parabola whose axis is a line parallel to the direction of the acceleration. From the point at which the combined motion commences lines are drawn to give the directions of the initial velocity and of the acceleration.

If the trajectory is that of a particle projected to give maximum range on a plane, the direction of motion of the particle at the point at which the particle strikes the plane is at right angles to the direction of projection. Thus by drawing the line which is both tangent to the curve and at right angles to the direction of the initial velocity we have at the point of contact of line and curve the point at which the trajectory strikes the plane.

This is shown in the accompanying sketch in which O denotes the point at which the combined motion began, OT the direction of the initial velocity and the trajectory as outlined. TP is the line which is both tangential to the curve and at right angles to the direction (OT) of projection, and it touches the curve at P. P will thus represent the point on the plane at which the particle strikes the plane, and while OP will represent the plane, YOY' represents the vertical through O or the direction of the acceleration. Measurement of the angles gives the angle TOY equal to the angle POT and shows that OT bisects the angle POY.

The velocity, representing the velocity of projection, can be varied by releasing the main-trolley from different positions so that the distance that it runs with the cross-trolley held back by the guider is different at each release. The main trolley assumes a uniform velocity at the same point on each path on each occasion, thus the distance which it runs under the influence of the falling mass previous to assuming the uniform velocity is varied from release to release, giving in consequence a different uniform velocity on each occasion. Combining these different velocities successively with the same acceleration, parabolas will be traced (Fig. 3) representing trajectories of

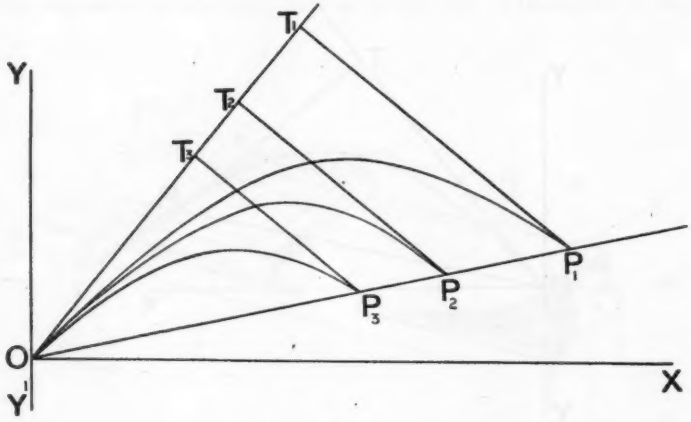


FIG. 3.

particles projected at the same angle with the horizontal but at different velocities. By drawing to each curve the tangent which is at right angles to the direction of projection, and obtaining thereby the point at which the

particle strikes the plane for which that angle of projection gives the maximum range, it is seen that P_1, P_2, P_3, the respective points of contact of each tangent with its curve and O, the point from which the particle was projected, lie on the same straight line OP. OP is then the plane for which that angle of projection gives the maximum range independent of the initial velocity. Thus is demonstrated the fact that although the velocity of projection may be varied the maximum range on a given plane for a given velocity is obtained by projecting the particle in a direction which bisects the angle the plane makes with the vertical.

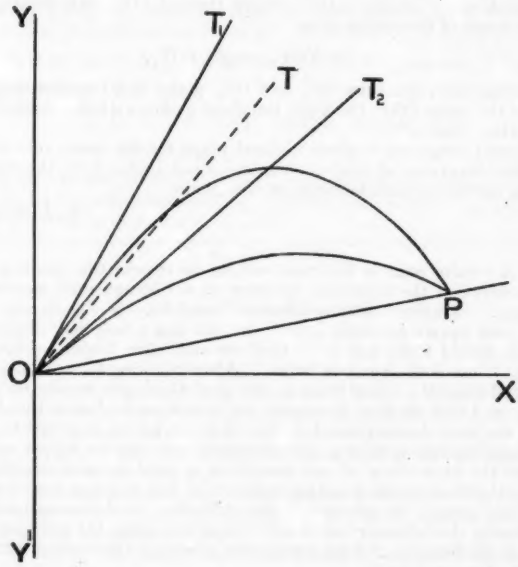
The truth of this statement may be demonstrated for any inclination of the plane by suitable arrangement of the angle between the directions of the velocity and the acceleration on the apparatus.

The other problem is to show that if a body be projected successively with the same velocity but at two different angles of projection, then for equal ranges on the same plane the directions of projections are equally inclined to the direction for maximum range, *i.e.* to the bisector of the angle made by the plane with the vertical.

For this demonstration it is found more convenient to take the direction of the velocity again along the direction of the main track and the direction of the acceleration (and hence of the vertical) along the direction of the cross track.

As the inclination of the velocity to the acceleration for the second trace is different from that for the first and it is the cross-track which is moved to effect this change, it will be necessary as seen later to move the paper between the registering of the motions.

Two scratches on the guider, one of which is at the end, indicate the distance through which the main trolley runs with an acceleration. With the accelera-



tion constant and the distance through which the actuating force is operating constant, the velocity of projection at each release is the same.

The cross-track is set at some suitable angle (40°) with the main track. The direction of the velocity, and the arrestment fork F_1 which operates for the main trolley, is adjusted to remove the actuating rider-weight when that trolley, as shown by the position of the load-post of the cross-trolley, is at the second scratch on the guider. The trolleys are now released from the first scratch, shown again by the position of the cross-trolley's load-post, and the trajectory is registered ((1) in Fig. 4). The point of projection of the particle is the point at which the combined motion begins to take place, namely, at the end of the guider. Through this point (O) as marked by the brush on the paper a line OY_1 is drawn, by releasing the cross-trolley to run freely along its track, to mark the direction of the acceleration, representing as it does the relative position of the vertical with respect to a similar motion in space. Also through O the direction OT_1 of the velocity is drawn.

To obtain the second trace the track is turned through an angle, say of 10° or 20°, to give a new angle of projection. This movement changes the direction of the acceleration, and hence of the represented vertical with regard to the paper. To adjust this the paper is turned and set so that (*a*) the point at which the combined motion starts, as shown by the point of the brush, is again at O, and (*b*) the direction of the acceleration through O coincides on the paper with the direction already marked as that of the represented vertical. As the brush has changed slightly its position with respect to the main trolley due to the movement of the track, the position of the arrestment fork (Fig. 1) must also be readjusted.

The second trace is now registered and found to intersect the former at P, giving OP as the plane upon which, for the same initial velocity, these angles of projection give equal ranges.

The direction OT_2 of the initial velocity through O is again drawn.

Measurement of the angles gives

$$\text{angle } YOT_1 = \text{angle } POT_2,$$

Thus showing that the lines OT_1 and OT_2 make equal angles with OT the bisector of the angle POY, the angle the plane makes with the vertical, thereby demonstrating that:

"For equal ranges on a given inclined plane for the same velocity of projection, the directions of projection make equal angles with the direction of projection for the maximum range on that plane."

E. J. ATKINSON.

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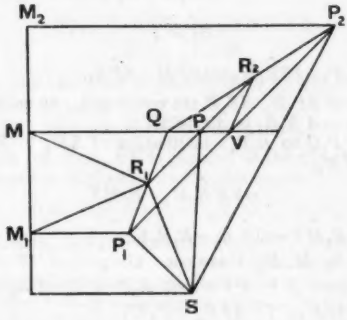
647. "A certain man in Kirmán, wishing to expose this ignorance [of the *mullás*], addressed the following question to a distinguished member of the local clergy. 'I agreed with a labourer,' said he, 'to dig in my garden a hole one yard square for eight *kráns*: he has dug a hole half a yard square. How much should I pay him?' 'Half the sum agreed upon, of course,' said the *mullá*, 'that is to say four *kráns*.' After thinking for a while, however, he corrected himself: 'two *kráns* is the sum which you legally owe him,' he declared; and this decision he committed to writing and sealed with his seal. Then the enquirer demonstrated to him that the labour required to excavate a hole measuring half a yard in each direction was only an eighth part of that needed for the excavation of one measuring a yard in each direction. This conclusion the cleric resisted as long as he could, but, being at length compelled to admit its justice, he got out of the difficulty by declaring that, though *mathematically* the labourer could only claim one *krán*, his *legal* due was two *kráns*."—E. G. Browne, *A Year amongst the Persians* (Cambridge, 1926, p. 507) [per Mr. F. Puryer White].

MATHEMATICAL NOTES.

924. [L¹. 10. a ; d.] *Note on the Geometry of the Parabola.*

[For brevity, noting that the straight line which bisects the join of two points A, B at right angles is the *radical axis* of the point circles A, B, this line shall be called the *axis of* A, B, or $Ax.\ (AB)$. Thus $Ax.\ (A, B)$ is the locus of points equidistant from A and B. Hence if P is on $Ax.\ (AB)$ and on $Ax.\ (AC)$ then it is on $Ax.\ (BC)$.]



P_1M_1, PM, P_2M_2 are perpendiculars on the directrix from three points P_1, P, P_2 on the parabola whose focus is S, such that $M_1M = MM_2$;

$\therefore$ P_1 is on $Ax.\ (SM_1)$, P is on $Ax.\ (SM)$, and P_2 on $Ax.\ (SM_2)$.

Let $Ax.\ (SM_1)$ meet PM in Q ;

$\therefore$ Q is on $Ax.\ (SM_1)$ and on $Ax.\ (M_1M_2)$;

$\therefore$ on $Ax.\ (SM_2)$,

i.e. Q is the circum-centre of the $\triangle SM_1M_2$ and QP_2 is $Ax.\ (SM_2)$.

Let R_1, R_2 be the mid-points of P_1Q, P_2Q and let QM meet P_1P_2 in V.

R_1 is on $Ax.\ (SM_1)$, also on $Ax.\ (M_1M)$;

$\therefore$ on $Ax.\ (SM)$.

So is R_2 ;

$\therefore$ R_1R_2 is $Ax.\ (SM)$ and passes through P ;

$\therefore$ $QP = PV$.

P_1P_2 is parallel to R_1R_2, *i.e.* P_1P_2 is parallel to $Ax.\ (SM)$.

Hence if M_1 and M_2 approach coincidence with M, keeping equidistant from it, the limiting position of P_1P_2 being the tangent at P, is $Ax.\ (SM)$.

Thus $Ax.\ (SM)$ is the tangent at P.

Similarly $Ax.\ (SM_1)$ and $Ax.\ (SM_2)$ are the tangents at P_1, P_2, and Q is the intersection of these tangents.

Hence the proposition that the chord joining two points P_1, P_2 on the parabola is bisected by the line parallel to the axis, passing through the intersection of the tangents at P_1P_2.

Also QV is cut by the parabola at its mid-point P, at which the tangent is parallel to P_1P_2.

Since PR_1 is $Ax.\ (SM)$;

$$\therefore\ P\hat{S}R_1 = P\hat{M}R_1 = R_1\hat{M}_1P_1 = R_1\hat{S}P_1, \text{ since } P_1R_1 \text{ is } Ax.\ (SM_1).$$

Thus two tangents from a point (R_1) subtend equal angles at the focus.

Again
$$\begin{aligned} S\hat{R}_1M_1 &= S\hat{R}_1M - M_1\hat{R}_1M \\ &= S\hat{R}_1M - P_1\hat{M}_1R_1 - P\hat{M}R_1 \\ &= S\hat{R}_1M - R_1\hat{S}P - P\hat{M}R_1 \\ &= S\hat{P}M; \end{aligned}$$

$$\therefore\ S\hat{R}_1P_1 = \tfrac{1}{2}S\hat{R}_1M_1 = \tfrac{1}{2}S\hat{P}M = S\hat{P}R_1.$$

Hence the triangles SP_1R_1, SR_1P are equiangular to each other.

P_1Q is $Ax.\ (SM_1)$ and R_1R_2 is $Ax.\ (SM)$;

$\therefore$ inclination of P_1Q to R_1R_2 = inclination of SM_1 to SM, or, since R_1 is circumcentre of SMM_1,

$$= \tfrac{1}{2}M\hat{R}_1M_1 = R_1\hat{M}V$$

$$\therefore\ R_1\hat{M}V = Q\hat{R}_1R_2 = R_1\hat{R}_2V;$$

$\therefore$ R_1, M, R_2, V are concyclic;

$\therefore$ rect. $MP\,.\,PV$ = rect. $R_1P\,.\,PR_2 = PR_2^2$;

$$\therefore\ 4SP\,.\,PV = 4PR_2^2 = P_2V^2.$$

The preceding supplies a proof of a fundamental property of the tangent, differing from that usually given ($P\hat{S}Z = 90°$), and also deduction of several well-known properties of the curve. R. F. M.

925. [L². 1. b.] *A Fallacy in Geometrical Conics.**

Since Dr. Bromwich has described (in the *Gazette* for last January) his experience with this fallacy, perhaps a little more space may be spared for the subject.

The explanation of the fallacy is perfectly simple. If S_1, S_2 tend to a point at infinity, while O_1, O_2 are fixed or tend to accessible points, the condition for O_1S_1/O_2S_2 to tend to unity is that the ratio of the projection of S_1S_2 on O_2S_2 to O_2S_2 tends to zero. In the vast majority of applications, S_1 and S_2 are the same point and the condition is satisfied automatically. Sometimes the distance between S_1 and S_2 is bounded, and then again no detailed examination is necessary. But when O_1S_1 and O_2S_2 are parallel chords of a hyperbola or parabola, the distance between S_1 and S_2 tends to infinity unless the chords themselves tend to coincide. Thus Newton's theorem cannot be applied without a preliminary investigation which is more troublesome than a direct attack on the problem.

But there is no need to have recourse to analysis, or to separate the parabola from the hyperbola. If a line through a point O cuts a conic in two accessible points P, Q and cuts the directrix corresponding to a focus S in Z, the line SZ cuts the eccentric circle of O in two points p, q, and we have

$$OP.OQ \,/\, pS.qS = ZO^2 \,/\, Zp.Zq,$$

whence
$$OP.OQ = \Pi\,/\,(1 - e^2\cos^2\theta),$$

where Π is the power of S for the eccentric circle, and θ is an angle which the line OPQ makes with the focal axis. Newton's theorem follows at once, since

* See Note 879, vol. 13, p. 422.

Π depends only on the position of O, and $1-e^2\cos^2\theta$ only on the direction of the line.

To find a theorem to replace Newton's when one of the points of intersection P, Q is at infinity, we have only to ask what relation there is in this case between the power Π and the position of the other point of intersection. If Q is at infinity, q coincides with Z. We have, as in general,

$$OP/ZO = pS/Zp;$$

the ratio on the right is equal to $\Pi / Zp.ZS$, and if U is the projection of O on ZS and V the projection of S on ZO,

$$Zp.ZS = 2\,ZU.ZS = 2\,ZV.ZO.$$

Thus
$$\Pi = 2\,ZV.OP,$$

and for two points and two pairs of parallel lines we have, in the circumstances presumed,

$$O_1R_1.O_1S_1 / Z_1V_1.O_1P_1 = O_2R_2.O_2S_2 / Z_2V_2.O_2P_2.$$

In the case of the parabola, SV is parallel to the directrix, and ZV is equal to XS for every position of O, whence for this curve

$$O_1R_1.O_1S_1 / O_1P_1 = O_2R_2.O_2S_2 / O_2P_2.$$

But for a hyperbola, Z_2V_2 is different from Z_1V_1 unless O_2 is on the line O_1P_1. In general, ZV is proportional to the distance of OP from the asymptote to which it is parallel: this is Dr. Bromwich's result.

Let us take the final step of supposing the line ZO to be an actual asymptote. Then P as well as Q is at infinity, p as well as q is at Z, and SZ is the tangent at Z to the eccentric circle. Hence the power of S for the circle is SZ^2, wherever O lies on the asymptote, and for parallel secants through any two points O_1, O_2 on the asymptote,

$$O_1R_1.O_1S_1 = O_2R_2.O_2S_2.$$

It is hardly an exaggeration to say that all the elementary relations between a hyperbola and its asymptotes, including the equation of the curve referred to the asymptotes as axes, follow at once from this last equality.

Whether a line through a point O making an angle θ with the focal axis cuts the curve or not, the quotient $\Pi / (1-e^2\cos^2\theta)$ exists unless the line cuts the curve at infinity, and this quotient plays for a line which does not cut the curve the part which the product $OP.OQ$ plays for an actual secant. By expressing theorems in terms of this quotient, which we may call the power of O along the line, it is possible to avoid encumbering enunciations and multiplying cases, while the return from power to product can always be made when the product is known to exist. Cartesian geometry in the real plane proceeds without discriminating between real and imaginary intersections of a line with a conic because the argument is conducted by means of quadratic equations, and the coefficients in these equations remain, whether or not the equations have roots. Pure geometry becomes equally fluent when instead of dealing with the distances OP, OQ we use a length and an area, determined from an arbitrary point O and an arbitrary line l through the point, which have the values $OP+OQ$ and $OP.OQ$ if the line does cut the conic. The area required is the power; the length is of course $2\,OW$, where W is the point in which l is cut by the diameter which bisects chords parallel to l.

E. H. N.

926. [K¹. 6. a.] *On the Introduction of Envelope Coordinates and Equations into Elementary Analytical Geometry.*

In the article on pp. 292 to 295 of the *Gazette* a preference is expressed in § 1 for envelope-trilinears over envelope-areals.

A point in favour of areals is the extra simplicity of the line at infinity, and teachers who have used Milne's *Homogeneous Coordinates* will probably agree in preferring them. Perhaps the choice is not very important. What *does* seem important is that the use of envelope coordinates and equations should be learnt concurrently with the corresponding locus work. The envelope coordinates chosen will be the ones that correspond to the locus coordinates that are used.

Actually non-homogeneous locus coordinates will occur before homogeneous ones; the corollary is that non-homogeneous envelope coordinates will also come first. These will not be the peculiar coordinates of § 13, but the coordinates corresponding to Cartesians, such as the coefficients (X, Y) of x and y in the equation $Xx+Yy+1=0$. Much of the elementary bookwork can profitably be set out in parallel columns, *e.g.*

There will be a discussion of the general (locus) equation of the second degree, viz.

$$s\equiv ax^2+2hxy+by^2+2gx+2fy+c=0,$$

including for example the determination of the condition for

$$Xx+Yy+1=0$$

to be a tangent-line of the locus (*i.e.* the envelope equation will be found).

Equations like

$$s=ks',\ s=k\alpha\beta,\ \alpha\beta=k\gamma^2,\ s=k\alpha^2$$

and

$$s=k\alpha,\ s=k$$

will be interpreted; so far as the last two are concerned this will be best done after the introduction of homogeneous coordinates in which $z=0$ is the line at infinity,

and so on.

There will be a discussion of the general (envelope) equation of the second degree, viz.

$$S\equiv AX^2+2HXY+BY^2+2GX+2FY+C=0,$$

including for example the determination of the condition for

$$xX+yY+1=0$$

to be a contact-point of the envelope (*i.e.* the locus equation will be found).

Equations like

$$S=kS',\ S=kAB,\ AB=k\Gamma^2,\ S=kA^2$$

and

$$S=kA,\ S=k$$

will be interpreted; so far as the last two are concerned this will be best done after the introduction of homogeneous coordinates in which $Z=0$ is the origin,

and so on.

The exceptional case $\Delta=0$ will require explanation such as that given on p. 290 of the *Gazette*, but will the explanation appeal to the intelligent student without the discussion of what is happening geometrically?

What has happened in the older text-books is that envelope coordinates have been introduced, at rather a late stage, and they have then been introduced in homogeneous form (l, m, n). In these books locus equations are often not made homogeneous at all, and this is probably because writers were willing either to avoid points at infinity altogether or to treat them in mid-Victorian fashion. In favour of the introduction of envelope coordinates first in the non-homogeneous form there are these two strong arguments: it helps to emphasise the idea of duality, and it forces the introduction of envelopes to an earlier stage in the teaching.

The work on p. 290 of the *Gazette* would be from this point of view part of the ordinary bookwork of $S=kS'$, which is specially important because of its application to foci and confocal conics. A. R.

927. [K1. 2. d.]. *The Radii of the Cosine and Lemoine Circles.*

In what follows, S denotes the circumcentre of any triangle ABC; K the symmedian point and centre of the cosine circle; L the mid-point of SK and centre of the Lemoine circle; R, ρ, λ the radius of the circumcircle, cosine circle and Lemoine circle respectively.

I cannot understand why in the text-books of Modern Geometry (in those, at least, that I have consulted) independent expressions are not given for ρ and λ. In Casey's *Sequel to Euclid* (1892), p. 198, and in Lachlan's *Modern Pure Geometry* (1893), p. 76, *e.g.* they are merely connected by the equation $\lambda^2=\frac{R^2+\rho^2}{4}$; *i.e.* each is made an explicit function of the other. But there is no difficulty in finding independent expressions for each of them.

Of course it is easily shown that

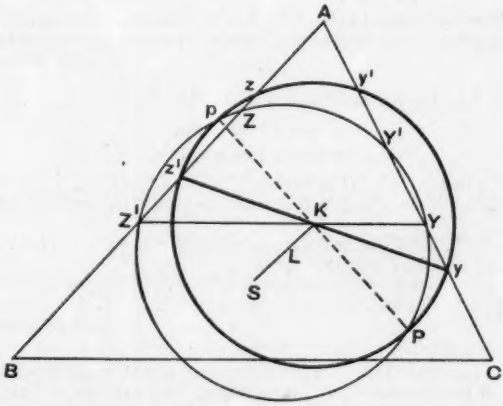
$$\rho=\frac{R}{\cot A+\cot B+\cot C}=R\tan\omega,$$

where ω denotes the Brocard angle of ABC; and consequently

$$\lambda=\frac{R}{2}\sqrt{(1+\tan^2\omega)}=\frac{R}{2}\sec\omega.$$

More practically convenient expressions, however, can be given in terms of R and SK.

For let the Lemoine circle cut CA in Y, Y', and AB in Z, Z'; and let YZ' be the chord through K that is parallel to BC; and let yz' be the diameter of the cosine circle that is antiparallel to BC, and consequently antiparallel to YZ'.



Then since the angles Yyz', $YZ'z'$ are equal, Y, y, Z', z' are concyclic;

$$\therefore\ YK\,.\,KZ'=Ky^2;$$

$\therefore$ K is a point on the radical axis of the two circles.

And since the radical axis is perpendicular to LK it must be the diameter of the cosine circle that is perpendicular to SK. It follows that the two circles intersect in the extremities of this diameter, Pp, say; and consequently

$$\lambda^2=\rho^2+LK^2=\rho^2+\frac{SK^2}{4}.$$

But $$\lambda^2=\frac{\rho^2+R^2}{4};\quad\therefore\ \rho^2=\frac{R^2-SK^2}{3}.$$

And it is easily deduced that $\lambda^2=\frac{R^2-SL^2}{3}=\frac{R^2-LK^2}{3}$.

More generally, if SK is divided in any point T in the ratio $l : m$, it is easily shown that the radius of the corresponding Tucker circle whose centre is T is

$$\frac{\sqrt{(l^2\rho^2 + m^2R^2)}}{l+m}.$$

Hence if τ denotes this radius,

$$\tau^2 = \frac{l^2+3m^2}{(l+m)^2}R^2 - \frac{ST^2}{3}.$$

It follows from the values of ρ and λ that if SK cuts the cosine circle in r, r', and the Lemoine circle in R, R', the vertex of either equilateral triangle described upon rr' or RR' is on the circumcircle. Hence we can readily construct a triangle, having given either (1) S, K, and a vertex, or (2) the circumcircle, the symmedian point, and a vertex of Brocard's first triangle.

We can also prove in a couple of lines by "mensuration" methods the well-known descriptive theorem that the radical axis of the circumcircle and Lemoine circle is the polar of K with respect to the Lemoine circle.

For let LK meet the polar of K in V; then

$$\begin{aligned} VL^2 - \lambda^2 &= (VS - LS)^2 - \lambda^2 = VS^2 - 2(VL + LS)\,LS + LS^2 - \lambda^2 \\ &= VS^2 - 2VL\,.\,KL - SL^2 - \lambda^2 = VS^2 - (3\lambda^2 + SL^2) \\ &= VS^2 - R^2. \end{aligned}$$

Thus V is on the radical axis of the two circles, etc. Conversely, of course, the expression for λ may be deduced from the theorem, proved otherwise.

G. WOTHERSPOON.

928. [X. 4. b.] *Geometrical Solution of the Equations.*

$$a\cos x + b\cos y = p,$$
$$a\sin x + b\sin y = q.$$

Construct a rectangle $OXO'Y$ having $OX = p$ and $O'X = q$.

With O, O' as centres and a, b as radii respectively, describe arcs cutting at P and Q.

Then $\angle$s XOP, XOQ are primitive solutions for x and $\angle$s $YO'P$, $YO'Q$ the corresponding primitive solutions for y.

If, in the equations, $-b$ is substituted for b, the angles at O' should be measured from $O'Y'$ (obtained by producing YO').

T. A. HONAN.
A. S. GOSSET TANNER.

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647. "As for the Table of Incomposits I was very sensible of the bad effects of perfunctoriness in Supputating, Transcribing, or Printing of it. My care therefore was not small: yet pag. 198, is almost filled with Errata, and I dare not warrant that none have escaped unseen: But seeing so few are fit to undertake to Supputate it anew, whosoever shall happen to discover any other fault in that Table, shall do well to signify it to the Book-seller, or to any other likely to be concerned in the next Impression."—Thomas Brancker, in the Preface to his Translation of Rhonius' *Algebra*, 1668. The original treatise contained a short list of primes, and as an Appendix to the Translation Brancker gave a table showing the smallest factor of every odd number less than 100,000 which is not a multiple of 5. But he explains that it was Dr. John Pell who taught him how to make the table! The Preface to the Translation, and his Table of Incomposits, are reprinted in Maseres' collection of *Tracts on the Doctrine of Permutations and Combinations*, pp. 353-416. The page of Errata referred to in the Preface is reprinted as p. 366, although the corrections are incorporated in the reprinted Table.

E. H. N.

REVIEWS.

Astronomy and Cosmogony. By SIR J. H. JEANS. Pp. x+420. 31s. 6d. 1928. (Cam. Univ. Press.)

It is more than a decade since the publication in 1917 of *Problems of Cosmogony and Stellar Dynamics* by the same author. That work was an essay that gained the Adams Prize at Cambridge; there are obvious limitations in the scope of such a publication; moreover, the eleven intervening years have been very fruitful of astronomical results. Readers will therefore welcome the author's decision to paint a picture on a more comprehensive scale than was possible in a Prize Essay. The new work, in fact, covers the whole field of stellar and nebular astronomy, and even includes the planets to the extent of considering theories of the manner of their formation. The book is mainly of a descriptive character; necessarily mathematical formulae and discussions cannot be wholly banished, but they do not occupy more than a tenth of the book; and since the results are given in plain language, the mathematics may be skipped without losing the thread of the argument. The opening chapters are an epitome of our present knowledge of the sizes and distances of the stars and nebulae; there are many homely illustrations to help readers to visualise the facts. Two of them may be quoted. Taking a full stop to represent the earth's orbit, then Alpha Centauri would be 75 yards away, and Hubble's furthest nebula a million miles away. On a still smaller scale the system of spiral nebulae is represented by 50 tons of biscuits spread uniformly through a sphere of a mile radius, each biscuit representing a universe some 12,000 light-years in diameter; our own universe, which seems to be unusually large, would be a flat cake 13 inches in diameter.

On page 34 we learn the immense range in the absolute brightness of the stars; about one star in two millions has ten thousand times the sun's luminosity; as we go down the scale the numbers rise rapidly; about a fortieth of the whole number have luminosities exceeding three times that of the sun; about a tenth of the whole are approximately equal to the sun; a decided majority are markedly below him in light-giving power; and one-third of the whole have only about one-thousandth of his luminosity.

We are next introduced to the distribution of energy in the spectrum and the effect of temperature upon it. In the case of the hottest stars the spectrum extends so far into the ultra-violet that our atmosphere will not let it all come through; on the other hand, in the case of the coolest red stars, a large part of the radiation is inappreciable to our eyes, being beyond the red end of the spectrum. A startling extreme case is afforded by the variable star χ Cygni; this changes visually from the fifth magnitude to the fourteenth; but Pettit and Nicholson found that the total radiation changes only in the ratio of 17 to 10; at minimum its radiation is almost entirely dark heat.

The modern change of view in the interpretation of stellar spectra is next explained; different spectra were formerly thought to indicate differences of chemical constitution; but it is now believed that the different types simply indicate different temperatures; so that if the sun's temperature were doubled, hydrogen lines would take the prominent place in its spectrum that the calcium lines do now. A combination of spectral data with the distances and magnitudes of the stars leads us on to "Giants" and "Dwarfs" and the "Main Sequence," terms that are now in every-day use; it is shown that this division of the stars was at first deduced from a small number of our nearer neighbours, but that it is amply confirmed by the much more extended material that is now available.

The next chapter discusses equilibrium in a star's interior under gravitation, gas pressure, and radiation pressure. This is of necessity a mathematical chapter; it leads to the conclusion that the density at the centre of many stars is so great that the state cannot be considered as gaseous, but is liquid. This is at present a matter of controversy, Prof. Eddington reaching a different conclusion. Chapter IV. is entitled "The source of stellar energy," and deals with a question of fundamental importance in astronomy. In the last century Lord Kelvin assumed that the sun's energy arose entirely from the coming together of its constituent particles from a great distance; he found on this

assumption that it could not have radiated at its present rate for more than 20 million years in the past; this was insufficient to satisfy the geologists, who demanded a period fifty times as long. The deadlock persisted till the discovery of radium led to a new conception of the atom and a realisation of the enormous amount of energy locked up in it. Thus Sir J. Jeans tells us that the annihilation of a single pound of coal would provide for the heating and lighting of this country for a century. He himself suggested annihilation of matter as the main source of stellar energy as far back as 1904. A less sensational suggestion was put forward by Perrin and Eddington about 1919; that if lighter atoms are transformed into heavier ones an amount of energy is liberated sufficient to give a duration for the sun that would amply satisfy the claims of the geologists. Jeans, however, gives arguments to show that this process would produce unstable stars. He therefore prefers the annihilation doctrine; he suggests that the atoms specially liable to it are of higher atomic weight than any known on earth, and that the supply of these gets exhausted in old stars, so that their output is very feeble: this view enables estimates to be made of the ages of different stars. Taking a million million years as the unit, the age of the faint red dwarfs is given as 200, of the sun as 7, of Sirius and Capella as 1, and of young giants like Betelgeux less than one-tenth. Jeans gives arguments to show that the conditions of annihilation do not depend on temperature, as Prof. H. N. Russell had suggested in 1925. The annihilation of matter would produce very short waves, but these would be lengthened in their passage from near the centre of a star to its surface. In the nebulae there would be less lengthening of waves; Jeans suggests that the very short "Millikan" rays arise from the annihilation of matter in the great nebulae.

There is a chapter, "Stable and unstable configurations," of which the substance had already appeared in *Monthly Notices of R.A.S.* The author traces the results of greater and greater ionisation of stellar atoms, and so obtains alternate stages of stability and instability in a star's career. A chart is given, constructed on the basis of this theory, showing the distribution of stars at various stages of their career as regards magnitude and spectral type. It agrees closely with the diagram of the observed distribution, which had been constructed by Prof. Russell.

The chapters on the figures produced by the rotation of liquid and gaseous masses are not greatly changed from the author's earlier volume; they have a bearing on three astronomical questions, (1) Laplace's nebular hypothesis, (2) the formation of binary stars by fission, (3) the forms of nebulae; in particular, the lens-shaped figures of some of these with a sharp equatorial edge; it is shown that in these objects a further increase in rotational speed would cause rings of matter to be thrown off at the equator, as Laplace asserted. Further examination, however, leads the author to the conclusion that Laplace's nebular theory of the origin of the planetary system is untenable, though it may explain some of the forms that we find among the nebulae.

A new result reached by the author as regards stellar rotation is that the passage of radiation through the star has a retardative effect on the rotation of the outer regions. He considers that the central core of the sun may rotate in three days or less; this core would thus be oblate and would approach nearer to the surface in the equatorial regions; it is suggested that the more rapid rotation at the sun's equator may be thus explained.

The origin of double stars is naturally explained by fission arising from rapid rotation; tidal friction will then produce a recession of the components to a moderate distance from each other. Since most of the visual binaries are too far apart for this explanation to be valid, the author concludes that the separation in these cases has been increased by the disturbing action of other stars which happened to pass near them. On this assumption he makes an estimate of the age of the stellar system (about 10^{13} years) which agrees closely with that already deduced by their output of energy. Incidentally, actual collisions of stars are found to be so excessively rare that they may be disregarded altogether.

Collisions of star with star are therefore excluded as an explanation of Novae, since on the average several Novae are discovered each year.

Approaches within a few million miles, such as are invoked on the planetesimal or tidal hypotheses of the birth of the planets, are much more frequent, but even these are estimated to occur only once in fifty million years, on which assumption only one star in 100,000 would be accompanied by a planetary system.

It is concluded from the paucity of very young stars that the birth-rate of stars reached a maximum at the time of the birth of stars now of absolute magnitude 2 to 5, and then declined so as to become almost negligible.

The chapter on the great nebulae embodies Hubble's recent results on their distances and sizes. The view of Sir William Herschel that these nebulae are "island universes" resembling our own star-system has undergone many vicissitudes in the intervening century, but is now fully re-established. The sizes of some of them are found to be comparable with, though not quite equal to, that of our own system, so that Shapley remarked, "If they are islands, this is a continent." Sir J. Jeans finds the nebular forms fall into the different classes which he had found from theory would be assumed by rotating masses of gas. The lenticular nebulae correspond to his Maclaurin spheroids; the barred spirals to his Jacobian ellipsoids, while the ordinary spirals exemplify the case where matter is shed outwards from the sharp edge of the lenticular forms. The emission of matter from two opposite regions of the central mass implies tidal forces; these may arise in a few cases from the proximity of another nebula, but more generally from the presence of adjacent matter, previously expelled from the nebula itself. Incidentally it is shown that the elliptical nebulae and the central regions of the spirals must be mainly gaseous, as star-clouds could hardly retain such regular mathematical forms; this gas is, however, condensing into stars, and the chains of luminous knots visible on the arms of the spirals are a stage in the process.

The shape of the spiral arms remains a puzzle; the equiangular spiral is a possible form of orbit under a force to the centre varying as the inverse cube of the distance; but it is impossible to imagine any distribution of matter that would give this law in all parts of the nebula. Prof. E. Brown suggested that the spiral arms might merely be the envelopes of orbits, not orbits themselves; but this suggestion was likewise found to require very complicated distribution of matter. In short, the enigma of the spirals has not yet been solved; the author ventures on the bizarre suggestion that in the central regions of spirals matter is being poured into our space from an extraneous dimension, and so apparently being created.

His views on the origin of the planetary system have been greatly modified since the publication of his earlier essay; the conclusion that it arose from the approach of another star, that raised great tides in the sun, remains unchanged. But he formerly thought that when this occurred the sun was so large as nearly to fill the present orbit of Neptune. His views on the duration of a star's life have been so greatly extended that he now assumes that the sun must have been practically of the same size and condition as now when the planets were born. Then, as now, it was greatly condensed towards the centre, and the filament that was to form the planets came from its outermost layers; the discussion goes further into details and concludes that Mercury and Venus became liquid immediately after birth, that the earth and Neptune were partly liquid and partly gaseous, while the other four planets remained gaseous during the birth of their satellites, which would account for the relatively small size of the latter.

The book is a remarkable one from the large number of difficult problems with which it deals, and the ingenuity that is shown in attacking them. At least two of the author's conclusions (the liquid condition of the interiors of certain stars and the fission theory of the Cepheid variables) are keenly controverted by some astronomers; but even these should welcome the full presentation of his arguments, which will aid in arriving at a decision.

ANDREW C. D. CROMMELIN.

(1) Sechsstellige Tafel der trigonometrischen Funktionen. By PROF. J. PETERS. Pp. viii + 293 (10·3 in. × 7·5 in.). 48 M. (Geb. 52 M.). 1928. (Dümmler, Berlin.)

(2) Sieben- und mehrstellige Tafeln der Kreis- und Hyperbelfunktionen und deren Produkte sowie der Gammafunktion. By K. HAYASHI. Pp. vi+283 (11 in. × 8·4 in.). 45 RM. (Geb. 48 RM.). 1926. (Springer, Berlin.)

(1) This is a very useful book for anyone who wants values of sin or cos to 6 decimal places, or of tan, sec, cosec, or cot to 6 (in some parts 7) significant figures. These values are given in Table II., the argument proceeding by interval 10″. For small values of x, Table I. gives cot x and cosec x to 4, 5, or 6 significant figures, and a supplementary table is provided by means of which either of these can be found to 6 significant figures. It is unfortunate that cot x precedes cosec x in Table I. and follows it in Table II.

The tables are beautifully printed, and it is practically impossible to overlook a change in the leading figures of a column.

Some proportional differences are given in the margins.

The values tabulated are stated to be correct within a little more than ½ in the final figure.

(2) Prof. Hayashi's book of tables is rather cumbrous; but it contains a good deal that is not to be found in ordinary books of tables.

The important table is Table I., which gives values of ϕ ($\equiv x/\pi \cdot 180°$) in ° ′ ″, $\sin x$, $\cos x$, $\mathrm{tg}\, x$, $\arcsin x$, $\arccos x$, $\mathrm{arc\,tg}\, x$, e^x, e^{-x}, $\mathfrak{Sin}\, x$ ($\equiv \sinh x$), $\mathfrak{Cos}\, x$ ($\equiv \cosh x$), $\mathfrak{Tg}\, x$ ($\equiv \tanh x$), $\mathfrak{Ar}\,\mathfrak{Sin}\, x$ ($\equiv \sinh^{-1} x$), $\mathfrak{Ar}\,\mathfrak{Cos}\, x$, $\mathfrak{Ar}\,\mathfrak{Tg}\, x$, $\log_e x$. The main part of the table is for $x = 1{\cdot}000$ to $2{\cdot}999$ and $3{\cdot}00$ to $9{\cdot}99$. The values are mostly given to 10 or more decimal places for the direct functions and to 7 or more for the inverse functions. The table also gives $\mathfrak{Amp}\, x$ ($\equiv \sin^{-1} \tanh x$) for $x = {\cdot}001$ to $2{\cdot}999$ and $3{\cdot}00$ to $10{\cdot}00$, to 7 places.

Table IV. gives the values of products of $\mathfrak{Sin}\, x$ and $\mathfrak{Cos}\, x$ by $\sin x$ and $\cos x$ to 8 places of decimals for $x = 0{\cdot}000$ to $10{\cdot}00$.

Tables VI. and VII. give $\log_{10} \Gamma(x)$ to 8 or more places for $x = 0{\cdot}00$ to $3{\cdot}00$ (interval ·01 to ·00001), and $\Gamma(x)$ to 7 or 8 places for $x = -5{\cdot}00$ to $5{\cdot}00$ (interval ·01 or ·001).

Following the above, or interspersed with them, are some tables of factorials, powers, etc. There is also an Appendix dealing with single interpolation—the method used is hardly up to date—and with formulae for subtabulation. The numerical values of coefficients in these formulae are given in minor tables (not mentioned in the contents list) inserted in vacant places in the volume.

The methods of calculating the tables are not stated, but apparently it is claimed that the tabulated values are correct within 1 in the final figure.

The arrangement of the tables is exasperating. The (incomplete) contents list says that Tables I., II., III., IV. commence on pp. 1, 14, 96, 205. The reader would naturally infer that Tables I., II., and III. occupy pp. 1-13, 14-95, and 96-204 respectively. What is not mentioned is that these tables end on pp. 204, 46, and 166. Table I., on pp. 1-204, is, as already stated, the main table. It is too wide to go on one page; so it occupies (broadly speaking) the upper seven-ninths of the left-hand and right-hand pages, with overflows (not always of the same function) into the lower two-ninths of the right-hand page and (sometimes) of the left-hand page. This would leave the lower portions of some of the left-hand pages empty; so they are used for the insertion of Tables II., III., and X., of some of the tables of coefficients ("Tabellen") mentioned in the Appendix (but not in the contents list), and of some bits of tables which have no headings and are not mentioned anywhere. These latter include: (p. 88) values of $\tan x$ near $x = \frac{1}{2}\pi$; (pp. 4-12) $\sin x$, $\cos x$, e^x, e^{-x} for very small values of x; and (p. 12) e^x, etc., for x = certain multiples of $\pi/4$.

On the same principle Table V., and a section from the Appendix, are inserted in vacant spaces of Table IV. (pp. 205-245).

As to the value of the tables, the following points should be noted:

(*a*) It is doubtful whether it is worth while giving so much space to the hyperbolic functions and their inverses. It may be assumed, for instance, that anyone who wants to find x to 10 figures when $\tanh x$ is given possesses a machine; and with the machine the value of $y \equiv e^{2x}$ is easily deduced from that of $\tanh x$, so that what we could do would be to find $\log_e y$ from y.

For the ordinary worker to 10 places of decimals, the important tables are those of e^x and $\log_e y$. (And 10 places are not always sufficient.)

(*b*) There is something incongruous in using one argument-column for entry of a function and of the inverse function. It is not only a matter of habit; the system of values in the argument-column which is suitable for the function may be quite unsuitable for the inverse function. In the present case, for instance, an early part of the argument-column has x from 0·0010 to 0·0999, the next part being from 0·100 to 0·999. Both of these are appropriate for the table of e^x; but they are quite inappropriate for that of $\log_e x$.

(*c*) It is a pity that the author does not follow the ordinary practice of entering on each page the line of entries which appears at the top of the next page. This practice is very helpful for checking, by differences, that there is no misprint.

(*d*) There is one very serious defect. Throughout the volume the author adopts the practice, for economy of paper, of printing one or more initial figures at the top of each column. But a change of these initial figures in the course of a column is very imperfectly shown. Only a small asterisk is inserted at the beginning of the entry (or, for a second or third change, two or three asterisks): this is apt to be overlooked by the unpractised eye. This method of arranging a table ought to have been obsolete long ago.

W. F. SHEPPARD.

1. **Collected Papers on Wave Mechanics.** By E. SCHRÖDINGER. Translated from second German edition. Pp. xiv+146. 25s. net. 1928. (Blackie & Son.)

2. **Selected Papers on Wave Mechanics.** By L. DE BROGLIE and L. BRILLOUIN. Authorised Translation by WINIFRED M. DEANS. Pp. viii+152. 15s. net. 1928. (Blackie & Son.)

3. **Four Lectures on Wave Mechanics.** By E. SCHRÖDINGER. Lectures given at the Royal Institution, London, March 5th, 7th, 12th, 14th, 1928. Pp. x+45. 5s. net. 1928. (Blackie & Son.)

We welcome most heartily these books on Wave Mechanics which have recently appeared, all bearing this simple title. No one of them contains exactly new matter, for the two larger are translations of collections of reprints of the original papers by Schrödinger and by Brillouin and de Broglie, and the other booklet contains the text of four lectures given by Schrödinger at the Royal Institution last spring. It is extremely valuable to have in a convenient form in English (and the translator's work appears to be admirable) the original papers of these authors, who have been primarily responsible for directing attention to the essential wave aspect of matter and mechanics, and for initiating and largely contributing to its rapid development. It should perhaps be mentioned that part of the contribution of Brillouin to the volume of which he is joint author (notably the first of his articles) is of a rather different nature, being concerned with summarizing the earlier advances in quantum mechanics over a wide field. This work was clearly only intended to serve as an introduction to the student to the original papers of Heisenberg, Born, Jordan, Dirac and others. It will still be useful for this purpose, but it is not intended for the profound study on its own merits which must be accorded to the other papers in these volumes. The account of the simple harmonic oscillator there given is also obviously wrong where it differs from the original account of Born and Jordan.

It will be agreed by everyone that the two bodies of doctrine usually known as wave mechanics and matrix mechanics are not rival theories but merely two aspects of the same theory, or even rather the natural expressions of two ways of thinking and talking about the same phenomena. Of these two the wave mechanics is from the physicists' point of view much the more novel. Physicists have long been accustomed to a particle theory of matter and to a wave theory of light—familiarity even with the latter extends back for a century. More recently, but still some twenty years ago, they had become familiar with the fact that light has also many of the properties of particles (the theory of

light quanta). But it is only recently, thanks to the authors of these books, that they realised that material particles have also many of the properties of waves. It is the recognition of this complete dualism which has enabled the striking advances of the last three years in atomic theory to be made. The point of view which finds its formal expression in the matrix machanics may be regarded as the natural extension and completion of the strict particle view of matter. The point of view expressed in wave mechanics is naturally based on an explicit recognition of the real novelty—the wave properties of particles. As a reaction to and a corrective for three hundred (or is it two thousand or more) years of particle prejudices, it is desirable that one should if anything over-emphasise the wave aspect of matter. One should at least study it deliberately in full detail, and for this purpose the papers of de Broglie and Schrödinger are essential. It is a fortunate circumstance that their study does not entail severe mathematical difficulties. The mathematical problems that arise are clear cut and of a familiar kind. If we do not wish to study the detailed solutions ourselves they are of a sort for which we can be easily and properly contented with a statement of the complete solution.

From every point of view therefore a study of the wave mechanics and particularly its origins as found in these books can be strongly urged on everyone interested, however remotely, in theoretical physics. One note only of warning need be sounded—a warning which is emphasized by Schrödinger himself in his introduction—that these papers being reprints of original papers written one by one do not as a whole represent a systematic development of the subject. They do not necessarily represent the point of view of the authors at any one time, still less now seeing that most of them were written before the end of 1926. But to avoid all misunderstanding it is only necessary to bear these conditions in mind. R. H. F.

Theory and Application of Infinite Series. By K. KNOPP. Translated from the second German edition by Miss R. C. YOUNG. Pp. xii + 572. 30s. net. 1928. (Blackie.)

This volume is the translation of Prof. Knopp's German treatise (1921 and 1923); the basis of the translation is the second edition (I have myself seen the first edition only, but the changes are not very numerous) together with a previously unpublished chapter (translated by Miss W. M. Deans) on what is called, in the older phraseology "*semi-convergent series*." For the actual work of translation itself, praise of high order may be given; it is not at all easy to detect any of the turns of thought by which one can usually notice the difference between German and English modes of expression. But I may take the opportunity of making a mild protest against the use of *monotone* as an *adjective*; this word is a well-established *noun*, in English, and appears also to be derived from a Greek noun; and, when the adverb *monotonely* is coined, there seems good reason for asking for a halt. I should also like to ask for some further consideration, before Analysis is finally committed to the use of the word *bound*, but I admit that no really good alternative has struck me, except the already overworked term *limit*.

Let us pass now to the actual contents—I assume that the majority of readers of the *Gazette* will not have seen the German original. The first chapter devotes some 40 crown octavo pages to the theory of irrationals, based on the Dedekind theory of sections. Here one may note the very convenient term "*nest of intervals*," introduced as equivalent to the German *Intervallenschachtelung*; this means a set of intervals of the type $(a_n b_n)$, where the ends a_n, b_n are both of monotonic type, the former *increasing*, the latter *decreasing*, and the length of the interval $(b_n - a_n)$ tends to zero. A simple and useful example is given by (J_n) where J_0 is the interval from 0 to 1, J_1 is the *left* half of J_0, J_2 is the *right* half of J_1, J_3 the *left* half of J_2, and so on, alternately. This particular nest shrinks up to the number $\frac{1}{3}$ (see p. 21); and in general a nest shrinks up to some unique number, rational or irrational.

The sequence

(a_n) is $0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}+\frac{1}{2^3}, \frac{1}{2}+\frac{1}{2^3}, \frac{1}{2}+\frac{1}{2^3}+\frac{1}{2^5}, \frac{1}{2}+\frac{1}{2^3}+\frac{1}{2^5}, \ldots$ and $(b_n - a_n)$ is $1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^2}, \ldots$.

The second chapter (of some 65 pages) deals with *sequences of real numbers*; here some account of *logarithms* is given, using the definition by

means of powers of a base; the reference to Napier (p. 56) implies that Napier used the base ordinarily denoted by $1/e$; it is therefore desirable to state once more that **Napier used no base at all, neither did Briggs.** Undoubtedly their logarithms *can* be defined in terms of those to bases $1/e$ and 10, respectively; but this idea of a base comes at a later stage in the historical development of the subject.

The next 150 pages are divided into six shorter chapters, dealing with the fundamental notions covered by the titles (slightly abbreviated in some cases) *Series of Positive terms*; *Series of Arbitrary terms*; *Power Series*; *Elementary Functions*; *Infinite Products*; *Sums of Series*. The following 200 pages are given over to four chapters, which expand the previous chapters to some extent; as, for example, the second chapter (Ch. IX.) on *Series of Positive terms*, contains the logarithmic scale of comparison tests and other detailed tests for convergence.

Chapter XIII. contains some 60 pages, in which a careful analysis is given of all the recent methods for associating a definite "*sum*" with a given "*divergent series*"; these are conveniently described by the initial letters of the various inventors' names; thus the *E*-process is associated with Euler, the *C*-process with Cesàro, the *H*-process with the arithmetic means of Frobenius and Hölder, while *B* and *R* refer to Borel and M. Riesz, respectively.

The final chapter of about 40 pages is the novelty of this edition, and should be found very valuable; but, in virtue of the influence exercised by Landau in Germany, and of the brilliant school of analysts who have grown up in England under the teaching of Hardy and Littlewood, it is natural to find special prominence given to those results which have arisen out of the investigations of the past twenty years on Prime-Number-Theory. I regret to see, however, that Heaviside's work on asymptotic series is apparently still unknown to the German school of analysts. When one is constantly using asymptotic series in the way Heaviside did, for numerical and operational solutions of wave-problems (electromagnetic and others), one is led to think that Heaviside was justified in expressing the views (*Electromagnetic Theory*, vol. ii. p. 10):

"Convergent mathematics is often excessively unpractical in the labour it involves in the numerical interpretation of results. I have noticed this particularly in spherical harmonic expansions, where the labour is sometimes prohibitive. Under these circumstances the substitution of equivalent divergent series which may be readily calculated becomes a matter of great practical importance. If the example fails, others may readily be given. But what about the theory of functions which deliberately ignores the treatment of divergent series? Can it be really *the* theory of functions? Is not a more comprehensive theory needed, including both convergent and divergent functions in a harmonious whole?"

T. J. I'A. Bromwich.

Les Espaces abstraits. By M. Fréchet. Pp. xi+296. 50 fr. 1928. (Gauthier-Villars.)

This book is intended as a source of references to all the many forms of "space" which have been invented within the last twenty years or so and as a discussion of the relations between them. A considerable fraction of the study is devoted to the more easily manipulated spaces defined in terms of a "distance." These spaces are not only of the greater practical importance. They also have properties more nearly parallel with those of Euclidean space of three dimensions. Even in this branch of the subject, in which Fréchet has been one of the important leaders, there is still room, apparently, for further discussion on many points.

In most readers, however, the second part of the book will rouse more interest because of the multitude of directions in which development can take place. It would take up too much space to enumerate the names of the various inventors, some of whom have been actuated by motives of pure interest, others by the desire for new tools to use in other fields. *En passant*, a noticeable gap is the lack of English names in the list. On the other hand, the sets of points [0], [*F*], etc., are mentioned on page 114 without any

reference being made to W. H. Young's valuable study of the *ul*, *lu*, *ulu* functions, etc.

To many readers, undoubtedly, this book will appeal most because of the many problems which are suggested and which have not so far been solved. There is one slight defect in that not many actual proofs are given. Fréchet contents himself on the whole with giving references to original papers. There is also—perhaps inevitably—a certain amount of jumping forwards and backwards which places the book in a different class from the more usual mathematical treatise. The subject may be compared more fairly with the Amazon basin than with a simple river like the Thames.

With all this the book is one to be strongly recommended to "nearly all" mathematicians—even to the mathematical physicist. One notes, for example, the work of Banach and Wiener on affine spaces generalizing Weyl's method of conceiving the problem of relativity. Moreover, there is one practical advantage of the subject in that it is easier in it than in some branches of mathematics for the intelligent reader to grasp what it is all about without five or ten toilsome years devoted specially to it.

The book contains a bibliography and index at the end, and, to sum up, it is a valuable addition to the Borel series of monographs. This, in itself, is high praise.

P. J. D.

Rise of Modern Physics. By H. Crew. Pp. xvi+356. 22s. 6d. n. 1928. (Bailliere, Tindall & Cox.)

The information in this book is large in amount, but it is not presented in the happiest manner, and is certainly not suitable for the elementary student. A history of any science should carry the reader from stage to stage in its development, always keeping his mind as far as possible in the line of thought developed in the period concerned. Comparison with later ideas should not be made until the later period is reached; otherwise the reader's mind is liable to approach a state of chaos, as when the present author in dealing with the work of Newton suddenly refers to J. J. Thomson and Southerns and then switches back to Huygens. Such treatment, in our opinion, makes it difficult even for the more advanced student to obtain a clear idea of the development which the author has set out to elucidate.

To attempt to cover the history of Physics in some 300 pages is obviously to strive for the impossible. One feels, however, that Galileo has more than his fair share in 14 pages, especially when it seems to be at the expense of Newton. Galileo was for his period a genius in virtue of the laws he developed as a result of experiment. Newton's great gift to science was the perfection of the Calculus and the application of equations to the representation of physical truths. In fact, Newton held much the same relation to Galileo as later on Clerk Maxwell held in respect to Faraday. The brilliant experimentalist was followed by the brilliant mathematician to correlate and develop his work.

The inequality of treatment becomes more pronounced when the tremendous amount of work of the last twenty or thirty years is reviewed in the comparatively small section of the book which is entitled "The Rise of *Modern* Physics," with hardly a reference to J. J. Thomson's pioneer work on the conduction of electricity through gases, from which current views of the structure of matter and of the universe have sprung. The author must sometimes have felt that to attempt the impossible is to cope with the inevitable. Looseness of expression, however, can and should be avoided. The statement on p. 15 that "this view . . . *elevated mathematics to a science* . . ." may on first reading be mis-interpreted. Also, surely, the fact that zero longitude passes through Greenwich is hardly an indication of international goodwill (p. 34), neither is the human eye a modern example of the camera obscura (p. 95).

On the whole the author has tackled a difficult thesis in an interesting manner, his information is in general sound, and he has obviously an exhaustive knowledge of his subject. The amount of work involved in its compilation doubtless merits the high price of the book, but this thought will probably be small compensation to the impoverished scientific reader.

B. M. Neville.

Introduction to the Mathematical Theory of Finance. By C. H. FORSYTH. Pp. v + 205. 12s. 6d. net. 1928. (Wiley: Chapman & Hall.)

As regards mathematics this is decidedly only an "introduction," there being very little algebraic argument involved; once the acturial symbols have become familiar the rest is easy. But the author's aim is to provide a training in direct approach to the practical problems, and in accordance with this more attention than usual is given to methods of approximating to actual conditions. Thus the question of re-investment of interest or of dividends is always kept to the fore and many interesting details of financial practice (American practice at any rate) are referred to. Beginning at once with simple and compound interest the usual subjects are handled; annuities, bonds, valuation, and depreciation, etc.; and at the end come two chapters on life and fire insurance. There are many exercises, a set of interest and commutation tables, and an appendix on iterative methods of solving equations as needed in determining rates of interest.

In this and the companion books on the Mathematical Theory of Life Insurance and the Mathematical Analysis of Statistics which have already been reviewed in the *Gazette* we certainly have a useful introduction to an important application of mathematics and one of better standard than most of the non-professional books. Apart from the American bias the chief fault of this book is a tendency to over-emphasis, as in the stress laid on the use of equations of value in very simple problems. The remark on page 3—"it will usually prove more natural to think of one dollar as the monetary unit, but it is an important fact that the mathematical theory of finance applies as well to any other monetary unit, such as the pound, franc, mark, etc."—leaves us wondering what kind of thing an exclusively "dollar" theory of finance would be.

G. SMEAL.

A First Course in Algebra. By W. G. BORCHARDT. 1928. Pp. xlvii + 260. n.p. (Rivington.)

Covers the ground as far as simultaneous quadratics in the competent way we associate with the name of Borchardt, includes a chapter on the Remainder Theorem.

1. Common-sense Algebra for Juniors. By F. F. POTTER and J. W. ROGERS. 2nd edition. 1928. Two parts. 130 and 120 pp. 2s. each. (Sir Isaac Pitman.)

2. Practical School Algebra. By C. V. DURELL. 1928. Pp. 320 + lx. 4s. and 4s. 6d., or in 3 parts. (G. Bell & Sons.)

These two books are singularly alike; they pursue the same aim, subject it to the same treatment, draw on much the same topics for illustration and question, and cover nearly the same ground, up to and including quadratic equations. The aim is to develop algebra as a generalisation of arithmetic not in the mechanical way of a generation ago but by copious induction. The examples draw on geometry and the properties of numbers, often anticipating results that belong to later stages of the subject, *e.g.*, early in each book S_n of $1+3+5+$ is found by generalisation from numerical cases, from a dissected square by the use of gnomons and (in Durell) by the use of the diagonals, $1+2+3+\ldots+n+\ldots+3+2+1$ being seen to be an equivalent sum. Geometry is freely used for illustration (Potter and Rogers use it to establish the method of completing the square) and for material for questions. Directed numbers and operations with them, including multiplication, are satisfactorily treated. Both approach the solution of simultaneous equations by showing graphically that it is necessary to find a common solution among the infinite solutions of the separate equations.

There are differences, of course: Potter and Rogers use an ordinary balance to illustrate the operations involved in solving simple equations (a rational proceeding when it is remembered that the original and concrete meaning of *æquus* is *level*), they deal with the gradient of a linear equation in x and y by a treads-and-risers method, they include some simultaneous quadratics with graphical illustrations and some simple partial fractions. Durell not only explains what to do but often issues warnings against common errors; he

includes the summation of common-difference series, and he uses freely a set of questions to teach a bit of book work.

Neither book deals with simultaneous equations in 3 unknowns, thus missing an opportunity of reinforcing and extending the idea of elimination. Neither book pays much attention to long division, etc.; these topics, too much neglected in modern books, would have found a suitable place in books whose motif is generalisation. Numbers expressed in the arabic notation would have appeared as special cases of algebraical functions, and the processes of multiplication, division, H.C.F. and square root for integers and decimal fractions would have been shown for algebra and arithmetic side by side.

It would be ungenerous to stress these omissions; it is more fitting to state as emphatically as possible that these books are crammed full of good things and are models of what elementary algebras should be. They are at once scholarly and practical; they appeal to the concrete in order to evoke a conception of the abstract. At every turn they *lead* to interesting discoveries, leaving to the pupil the joy of achieving them. They are essentially stimulating, and if anyone, pupil or teacher, mathematician or not, can use them without attaining the conviction that the study of algebra is a joyous adventure, he is beyond hope of mathematical salvation.

Elementary Algebra for Schools: Part I. By R. C. FAWDRY and H. C. BEAVEN. Pp. 220 + xxxvi. 3s. and 3s. 6d. 1928. (A. & C. Black.)

This follows the same general trend as other recent elementary algebras, *i.e.* it treats algebra as generalised arithmetic. As one would consider proper to such a method, literal equations are introduced early. The early chapters deal with the formula as (1) a generalisation of known particular cases, (2) a given general statement from which particular cases can be obtained. The products $a \times 0$ and $4 \times (-3)$ are obtained by appealing to the law $x \times y = y \times x$. Algebraical terms and operations are carefully explained, and those hints and cautions which the pupil gets from an experienced instructor are printed in the text. There are good historical notes.

The ground covered includes some simultaneous quadratics, the Remainder Theorem and Cyclic-Order fractions.

Brief references to "degree," "linear function," "symmetrical in a, b and c," seem to indicate an intention to go deeper into algebraic form than is usual at this stage. But the intention is not carried out. To degree, at any rate, more attention could easily and usefully be given in elementary work than is usually done.

The get up of the book is in every way attractive.

College Algebra. By C. I. PALMER and W. L. MISER. Pp. xiv + 377. 12s. 6d. 1928. (McGraw-Hill.)

Evidently American, "written for use in the freshman year of colleges and technical schools" this text-book does not seem suited for any purpose in English education.

Its get-up is most attractive, it has plenty of exercises and it gives good historical notes. Here our commendation ends. The book revises the solution of simple equations, introducing determinants into the solution of simultaneous groups and proceeds to topics usually classified as higher algebra. These include the Binomial Theorem (proved by induction for positive integral indices), Theory of Equations including Horner's method, Convergence but no Summation of Series, Complex Number and Determinants.

In the introduction we read "In its preparation one thought kept constantly in mind is: The student should feel that he is dealing with a subject that has real life connections." In vain we look for evidence that this thought is kept in mind at all; indeed, in the first chapter the authors set out apparently to present the subject as an abstract science and as far as the book can be said to pursue any idea, this is the one that emerges. But it is pursued with a painstaking eagerness that is pathetic in its failure.

The first chapter deals with fundamental ideas, operations and laws. The latter are introduced thus:

"It is assumed that there is an operation, or process, called addition, of

combining two numbers. If two real numbers a and b are given, we assume the following :

(1) *There is one and only one number s such that $a+b=s$.*

Since no distinction is made between a and b, we can as well say $b+a=s$. This gives the following, known as the **Commutative Law** for addition.

(2) $a+b=b+a$."

The opportunity to relate the commutative laws to our general experience of concrete magnitudes is neglected and the "real life connection" lost. We confess to having felt that the above form of words invited us to put out of our minds any preconceived ideas we had of the nature of addition and to accept the statement as a fundamental axiom of algebra. But that was only the beginning of perplexities and we found ourselves wondering how the authors would distinguish between addition and multiplication and why this form of argument should be applied to $a+b=b+a$ and to $ab=ba$ (treated in the same ways by the authors) and not to $a^b=b^a$. The laws once stated are applied to establishing $0.a=0$, $a(-b)=-ab$ and $(-a)(-b)=ab$ where a and b are real numbers. But after the first sixteen pages they are forgotten. In the treatment of complex numbers the operations of addition and multiplication are illustrated by the Argand diagram, but the fundamental laws are not mentioned.

Concerning indices we read : "Nothing compels us to bring into mathematics exponents that are not positive integers, but it is found very convenient to do so. The question then arises as to the meaning of exponents that are fractional, etc. Of course, these new symbols can be defined at pleasure ; but, for convenience it is important that all exponents should be governed by the same laws."

Further comment seems superfluous. F. C. B.

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649. At Chambord, between Orleans and Blois, is a magnificent Renaissance palace, built in 1526 for Francis I. It contains a famous staircase, which attracted the attention of seventeenth-century travellers. Evelyn says of it (*Diary*, 28th April, 1644, Globe ed. p. 44) : "The staircase is devised with four entries, or ascents, which cross one another, so that though four persons meet, they never come in sight, but by small loop-holes, till they land. It consists of 274 steps (as I remember), and is an extraordinary work, but of far greater expense than use or beauty." Francis Mortoft, whose travel diary has recently been edited by Malcolm Letts (Hakluyt Society, 1925), visited it on 20th September, 1658, and noted (p. 8) : "One may mount up there by a double pair of staires, which are so large that those that mount up together may speake to one another without being seene. There are about 274 degrees. Being at the Topp, one may cast a Ball or apple straight downwards through a little hole which appears belowe."

Query, what is the geometrical description of the staircase ? A recent book, *Old Touraine*, by T. A. Cook (2 vols. London, 1892, and later editions), does not, I think, make it very clear (ii. p. 193) : "It has two openings, and by imagining two huge corkscrews one within the other, whose curves ascend together yet never touch except at their extreme edges, the perplexed visitor strives to understand how it comes about that his companion, who is mounting upward like himself, can never meet him though never be completely lost." Have the torsions of the two helices the same or opposite signs ?—[Per Mr. F. Puryer White.]

650. The quantities which are dashed thus $\not{x}$, are such as destroy one another.—*Sir Isaac Newton's Two Treatises of The Quadrature of Curves, and Analysis by Equations of an infinite Number of Terms, explained.*—By John Stewart, 1745, p. 406.

651. Tel jour, Poisson cessa de professer et de vivre.—Arago, *Oeuvres* 1854, vol. 2, p. 658.

652. Le 7 Septembre, 1783 . . . Euler cessa de calculer et de vivre. Condorcet, *Eloge de M. Euler* (Oeuvres de Condorcet, edited by Arago, 1847, iii., p. 41).

MATHEMATICAL ASSOCIATION.

(LONDON BRANCH.)

PROFESSOR ALFRED LODGE delivered his Presidential Address on 2nd February at Bedford College, Regent's Park, on "Dimensions and Identity of Vector Direction." Starting with the applications of Dimensions (or Units) and of Vectors to problems in Mechanics, he extended their use to the whole domain of elementary school mathematics, giving illustrations of a novel and illuminating character in connection with arithmetic, Pythagoras' Theorem, a graphical solution of $ax^2+bx+c=0$, the tracing of the straight line $ax+by+c=0$, the range of a projectile, and partial differentiation. In every case he emphasised the importance and value of interpreting formulae through these two notions. Messrs. Katz and Sandon also spoke.

THE LIBRARY.

160 CASTLE HILL, READING.

The Librarian reports the following gifts:

From Mr. C. PENDLEBURY, his book:

Lenses and Systems of Lenses, treated after the manner of Gauss - - - - - - - 1884

From Miss F. A. YELDHAM copies of her papers on mathematical history, and of her book:

Story of Reckoning in the Middle Age - - - 1926

The following have been bought:

P. DE FERMAT Œuvres; IV - - - - - - - - - - 1912

The definitive edition, prepared by P. Tannery and C. Henry. The Library has already the first three volumes, and would welcome the fifth volume to complete the set.
Rouse Ball, in footnote to his *Short History*, 1901 edition, gives vol. iv. 1901, but speaks only of three vols. in the 1908 edition.
Vol. iv. is dated 1912, and gives no hint of having been expected at an earlier date, nor any hint of a future fifth volume, nor any explanation of the long delay from 1896, the date of vol. iii.
Evidently vol. iv. was expected so confidently by 1901 that Rouse Ball in this edition of his *Short History* which came out in that year took the risk of referring to the volume as though it already existed and bore that date: the lesson of bibliographical caution was not lost, and in the next edition there is no mention of a volume still to come.

C. STURM Auflösung über der numerischen Gleichungen - 1904
1835 Ostwald 143

Translated from French and edited by A. Loewy.

Ueber die Anziehung homogener Ellipsoïde - - 1890
Ostwald 19

Memoirs by Laplace (1782), Ivory (1809), Gauss (1813), Chasles (1838), and Dirichlet (1839) translated and edited by A. Wangerin.

G. H. BRYAN 1864–1928 **M. J. M. HILL** 1856–1929

ERRATUM.

The Gleaning numbered 640 is really the two opening lines of 639.

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD.
THE UNIVERSITY PRESS, GLASGOW.

MATHEMATICAL ASSOCIATION.

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LONDON BRANCH.

PROGRAMME FOR 1928-29.

All meetings will be held at Bedford College, Regent's Park.
They will be on Saturday afternoons, and will begin at 3 p.m.

1929.

Feb. 2nd. Presidential Address: "Dimensions and Identity of Vector Direction."—Professor ALFRED LODGE.

Feb. 23rd. Annual Business Meeting.
Discussion of Members' Questions.

Mar. 16th. Debate on Standard Form.
(Speakers: Miss PUNNETT and A. S. GRANT.)

The Secretaries will be glad to receive:

(i) Topics for discussion on February 23rd.
(ii) Names of Members willing to take part in the Debate on March 16th.

THE MATHEMATICAL ASSOCIATION.

(An Association of Teachers and Students of Elementary Mathematics.)

"I hold every man a debtor to his profession: from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves, by way of amends, to be a help and an ornament thereunto."—BACON (Preface, *Maxims of Law*).

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THE MATHEMATICAL ASSOCIATION, which was founded in 1871, as the *Association for the Improvement of Geometrical Teaching*, aims not only at the promotion of its original object, but at bringing within its purview all branches of elementary mathematics.

Its purpose is to form a strong combination of all persons who are interested in promoting good methods of teaching mathematics. The Association has already been largely successful in this direction. It has become a recognised authority in its own department, and is continuing to exert an important influence on methods of examination.

The Annual Meeting of the Association is held in January. Other Meetings are held when desired. At these Meetings papers on elementary mathematics are read and discussed.

Branches of the Association have been formed in London, Bangor, Yorkshire, Bristol, Manchester, Cardiff, the Midlands (Birmingham), New South Wales (Sydney), Queensland (Brisbane), and Victoria (Melbourne). Further information concerning these branches can be obtained from the Honorary Secretaries of the Association.

"**The Mathematical Gazette**" (published by Messrs. G. BELL & SONS, LTD.) is the organ of the Association. It is issued at least six times a year. The price per copy (to non-members) is usually 2s. 6d. each. The *Gazette* contains—

(1) ARTICLES, mainly on subjects within the scope of elementary mathematics;

(2) NOTES, generally with reference to shorter and more elegant methods than those in current text-books;

(3) REVIEWS, written when possible by men of eminence in the subject of which they treat. They deal with the more important English and Foreign publications, and their aim is to dwell on the general development of the subject, as well as upon the part played therein by the book under notice;

(4) QUERIES AND ANSWERS, on mathematical topics of a general character.